Resources for Quantum Information Tasks
From the bipartite to the multipartite scenario

PhD thesis
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Abstract

Quantum information theory shows that encoding information in quantum systems achieves results unattainable by any classical means. The aim of the present thesis is to contribute to the identification and characterization of quantum entanglement and quantum nonlocality, the resources believed to underlie the power of quantum physics at information-processing tasks.

We address the question of designing better direct methods of entanglement detection. For that, we study physical approximations to positive (non-physical) maps, which are used in the most efficient criterion of identification of entangled quantum states. In all considered cases, physical approximations to optimal positive maps define entanglement-breaking channels. These channels are useless for entanglement distribution and can be replaced by measurement and state-preparation protocols. Hence, physical approximations to optimal positive maps have a simple practical implementation and might contribute for improved entanglement-detection schemes.

Concerning the characterization of quantum nonlocality, we start by considering the bipartite scenario. We study the relation between nonlocality, entanglement and noise. In particular, we study the robustness of nonlocal correlations under the action of depolarizing noise, in a regime where entanglement is still present. We show that, for a generic class of quantum states, noise completely destroys the nonlocal properties of states, before entanglement is also lost. More specifically, we provide bounds on the amount of noise that transforms nonlocal entangled quantum states into local entangled quantum states. This means that local entanglement can be found frequently, and is not a particular property of highly symmetric mixed states.

Then we move into the multipartite scenario, where quantum correlations are shared by more than two subsystems. We define the following multipartite strategy for the study of nonlocality. Local measurements are performed on a subset of the parties, which leave the remaining ones in given quantum state. If this state violates a Bell inequality, it implies that the original state (held by all the parties) necessarily contains nonlocal correlations. This procedure reveals itself very useful for the study of nonlocality in a broad sense. First, we prove that there exist multipartite quantum states, pure and mixed, which contain fully-nonlocal multipartite correlations. This consists of the strongest form of genuine multipartite nonlocality. Then, we take copies of bipartite states and distribute them by several parties, in such a way that the whole system is in some multipartite state. We are then able to apply our multipartite strategy. Our second result is a direct link between quantum non-locality and one-way entanglement distillability. And third, we prove that nonlocal resources can be activated. Performing collective local measurements on \( N \) copies of a bipartite
local quantum state reveals its bipartite nonlocal content. Analogously, there exist genuine multipartite quantum states composed by $N$ copies of some quantum state that individually does not contain any genuine multipartite nonlocality.

Finally, we characterize quantum correlations by distinguishing them from stronger-than-quantum correlations. We define the first (non-trivial) multipartite nonlocal game, called ‘Guess your neighbour’s input’, for which quantum resources offer no advantage over classical ones. We show that supra-quantum resources are, sometimes, able to perform better at this game. Our results then suggest that physical multipartite correlations might be limited by a generalized no-signaling principle.
List of publications


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Chapter 1

Introduction

Quantum theory is the model that better describes the very small-scale Physics. Despite this evidence, the non-intuitive formulation and predictions of the quantum model made it particularly hard to be accepted by the general scientific community. Quantum theory gives priority to the pragmatic explanation of the observed physical phenomena. Even if this requires abandoning principles that might seem, at first sight, natural and essential in any theory of Nature. For instance, quantum theory abandons determinism: in general, observables do not have a definite value prior to the measurement process. It also predicts the impossibility of knowing, simultaneously, the exact value of some observables.

Other remarkable quantum features are (i) the phenomenon of quantum entanglement, that is, the existence of pure quantum states of composed systems that cannot be factorized into product states, and (ii) the existence of nonlocal correlations between the outcomes of local measurements, performed on separated systems sharing an entangled quantum state. These phenomena are the essence of a different way of dealing with the quantum model. Nearly twenty years ago, its strangeness started being explored as a powerful resource for information-processing tasks. It has been realized that it achieves impressive results, unattained by any classical method. For example, take perfectly secure (quantum) cryptography [BB84, Eke91], perfect (quantum) teleportation [BBC+93], and prime factorization in polynomial time [Sho97]. But these landmarks are only a small sample of the new range of possibilities offered by encoding information in quantum systems [NC00, HHHH09]. Quantum Information Theory emerged then as the study of this new interdisciplinary field between physics and information theory.

Although having experienced great advances, quantum information theory is still unable to provide an answer to the basic question: why do quantum resources provide an advantage over classical ones? Also, the search for efficient methods to identify, characterize and quantify these resources still continues. In the process, such operational approach to quantum theory gave new insight on its foundations. Can the interdisciplinary link between quantum and information theories finally provide intuitive principles that replace the standard axioms upon which the quantum model is based on?
CHAPTER 1. INTRODUCTION

1.1 Motivation

The main motivation of this thesis is to contribute to the identification and characterization of quantum entanglement and quantum nonlocality. Here, the general approach is to compare these with classical and supra-quantum resources, and to explore the multipartite framework. In return, this is expected to provide a deeper understanding on the structure of quantum theory. Inside such a vast topic, rich in answered questions, here the focus goes on the following.

Are there better methods for entanglement detection? Quantum entanglement is used in a considerable amount of quantum protocols, and it is therefore important to design efficient experimental methods to detect entangled quantum states. From a theoretical perspective, there exist two necessary and sufficient conditions for the presence of entanglement: one based on the expectation values of a given class of observables – the entanglement witnesses –, and the other based on the action of non-physical maps – the positive maps – on the density matrix that describes the quantum state.

The first method has the advantage of being directly implementable in the laboratory, while the second requires prior state-estimation. This process becomes increasingly hard with the size of the system, which is a serious drawback in its implementation. On the other hand, detecting entanglement with positive maps is much more efficient than with entanglement witnesses, in the sense that a single positive map is able to identify a much broader class of entangled states than a single witness. Then, an interesting possibility would be to combine the advantages of each method. Can we find efficient approximate physical realizations of positive maps, which can improve entanglement detection schemes?

How robust is nonlocality under the presence of noise? Quantum entanglement is not the only resource used in quantum processing tasks. In fact, there are cases where it is known not to be enough, and a stronger resource is required: quantum nonlocality. Consider, for instance, quantum communication complexity problems [Bra01, BZZ02], device-independent quantum key distribution [ABG+07, AGM06, BHK05], or randomness generation [PAnM+10]. In all these examples it is necessary that the outcome correlations between the outcomes of local measurements on separated subsystems, sharing an entangled quantum state, could not be obtained by any classical method. This is the same as saying that the outcome correlations must be nonlocal. For pure quantum states, entanglement is synonymous with nonlocality [Gis91, PR92]. Surprisingly, in the mixed-state case, there exist quantum entangled states which are also local [Wer89, Bar02]. These entangled states are then useless for some tasks, and it is therefore relevant to correctly identify them. In particular, it is important to understand if the action of noisy environments over pure entangled quantum states can destroy their nonlocal properties, while entanglement is still present.

How to characterize multipartite quantum nonlocal correlations? Quantum nonlocal correlations are much better understood in the simplest bipartite scenario. It is natural, however, to expect that multipartite correlations offer a much more valuable resource. They can even turn out be necessary, for
instance, for multiparty quantum information tasks. A proper understanding of this scenario is further motivated by the fact that highly correlated many-body quantum systems are natural sources of multipartite quantum states. A particularly interesting problem is the characterization of genuine multipartite nonlocal correlations, those that simultaneously involve all subsystems sharing a multipartite quantum state. However, very little is known about this subject beyond the most basic results. Genuine multipartite nonlocality was defined [Sve87, SS02, CGP+02] and there exist Bell inequalities able to detect genuine entangled quantum states [Sve87, SS02, CGP+02, JLM05, BBGP09]. The extension of most results on bipartite nonlocality to the multipartite framework is, indeed, still missing.

**Is it possible to activate nonlocal correlations?** Coming back to the relation between entanglement and nonlocality, it is certainly puzzling the existence of quantum entangled states which do not manifest any nonlocality. Following this idea, several generalizations of the standard tests for nonlocality have been proposed. Allowing sequential measurements [Pop95] / local filtering [Gis96], collective local measurements [Per06a], and/or the interaction with other local resources [MLD08], has revealed “hidden-nonlocality” in entangled quantum states local under single measurements.

Particularly interesting is the case where sequential measurements are not allowed. Is it possible to activate nonlocal resources by taking many copies of it, eventually with the help of local ancillas? The activation of quantum nonlocality, or non-additivity in a slightly weaker form, is an important feature from a practical perspective. It implies that the combined use of two quantum objects, for accomplishing a given information task, is more efficient than the two objects used independently. Recently discovered examples of supra-additive quantum resources are (i) the capacity for a quantum channel to transmit quantum, classical, and private information [SY08, Has09, LWZG09] and (ii) the entanglement of formation and the distillable entanglement [Has09, SST03]. It is then expected that quantum nonlocal correlations can be activated, which remains as an open question.

**What are the limits of multipartite quantum information resources?** Finally, we can perform an alternative characterization of quantum resources. Instead of studying situations where they beat classical resources, it is interesting to consider information processing tasks for which quantum resources are beaten by general nonlocal ones. This different approach is particularly important since it is not known which physical principle(s) impede quantum correlations from being stronger. In fact, imposing the no-signaling principle, i.e. the impossibility of arbitrarily fast communication, still allows for the existence of supra-quantum correlations [PR94, BLM+05].

The interplay between quantum concepts and information theory has shown that the existence of such stronger-than-quantum correlations would have deep consequences at the level of information concepts. They would, for instance, collapse communication complexity [vD05, BBL+06, BS09] and allow perfect nonlocal computation [LP SW07]. In a related direction, it has been proven that quantum correlations obey a strengthened version of no-signalling, the principle of information causality [PloPK+09]. This suggests the possibility that the
limitation of quantum theory, and therefore its foundations, can actually be understood in a more natural way by imposing basic principles involving concepts from information theory. Up to now, all known results on this subject refer exclusively to the bipartite scenario.

1.2 Main Results

In this thesis, we provide answers to some of the open questions mentioned before.

Relation between physical approximations to positive maps and entanglement breaking channels

Structural approximations to positive (non-completely-positive) maps have been introduced as approximate physical realizations of these non-physical maps [Hor03]. We study the implementation properties of structural approximations to positive maps, which can find applications in the design of better entanglement identification schemes. We observe that many of these approximations correspond to entanglement-breaking channels, which are useless for entanglement distribution. Consequently, these physical approximations can be prepared via measurement and state-preparation protocols, and might lead to simpler entanglement detection methods based in the positive map criterion. We apply this method to the well-known PPT (positive after partial transposition) criterion. Our results prove that it is possible to significantly simplify the proposal of [HE02], by substituting parts of the quantum protocol by classical processing. In parallel, these findings shed new light on the structure of the sets of quantum states and entanglement witnesses.

Noise robustness of quantum nonlocality

We consider bipartite entangled states, composed by the mixture of pure maximally-entangled states and local noise. We are able to construct local models for projective and general local measurements acting on them, which proves that they are local while still entangled. We then extend our results to mixtures with any quantum states. This method provides bounds on the noise resistance of the nonlocal correlations of completely general bipartite quantum states. We observe that nonlocality is clearly less noise-robust than entanglement. Indeed, for projective measurements, the critical noise parameter for which noisy states admit a local description is, at least, asymptotically $\log(d)$ larger than the critical value for separability.

A multipartite strategy for the study of bipartite and multipartite nonlocality

We present a strategy to study the nonlocality for multipartite quantum systems. It is based on performing local measurements on a subset of the parties and studying the nonlocal properties of the state of the remaining parties. This provides sufficient conditions for a quantum system to be fully-nonlocal according to a given partition as well as (genuinely) multipartite fully-nonlocal. Quantum full nonlocality, a phenomenon present in the bipartite case, is then also observed in genuine multipartite nonlocal correlations. In particular, we identify all completely-connected graph states as multipartite fully-nonlocal quantum states. Moreover, also give an example of fully genuinely multipartite mixed quantum state.
1.3. OUTLINE OF THE THESIS

This strategy also provides interesting results at the level of bipartite nonlocality. For that, we distribute copies of bipartite states over several parties, according to a given configuration, forming a multipartite state. We are then able to establish a direct link between one-way distillability and quantum bipartite nonlocality. Any one-way distillable quantum state displays some nonlocality.

**Activation of nonlocality in a multipartite scenario**  The activation of nonlocality is the last achievement of our multipartite strategy. This is obtained by disposing copies of bipartite quantum states according to a new geometrical configuration. We show examples of composed quantum states that display nonlocal features not observed for each individual state. We prove that it occurs both for bipartite and genuine multipartite nonlocality.

**Multipartite nonlocal game with no quantum advantage**  We present the first multipartite nonlocal game for which (i) quantum correlations never perform better than classical ones but (ii) general no-signaling correlations can provide an advantage. Our study involves identifying the winning probability with the violation of a Bell inequality. Interestingly, some of these Bell inequalities correspond to facets of the local polytope. The nonlocal game captures then intrinsic classical/quantum properties. Our results suggest that quantum correlations might obey a generalization of the usual no-signaling conditions in a multipartite setting.

1.3 Outline of the thesis

The present thesis is organized as follows. In chapter 2, I introduce the background material supporting the findings of this work. It is divided according to the two main resources, quantum entanglement and quantum nonlocality. In the first part, I introduce quantum entanglement as well as necessary and sufficient criteria for its detection. Then I review the role of completely positive maps in describing quantum processes and the properties of particular quantum channels. I also introduce the idea of physically approximating non-physical processes. In the second part, I include a general background on quantum nonlocality. I discuss its relation with entanglement, the problem of its activation by means of more general tests of nonlocality, and its generalization to the multipartite framework. Finally, I consider general nonlocal correlations and present the most important results concerning stronger-than-quantum nonlocal correlations. In chapters 4-7, I present the work developed during this thesis and the obtained results. Finally, in chapter 8, I present a general overview of the thesis and future perspectives. In appendix A, I introduce the geometry of the sets of local, quantum and no-signaling correlations. The remaining appendices constitute supplementary material for some of the chapters containing new results.
Chapter 2

Background

Entanglement is possibly the most basic concept in quantum information theory. The fact that quantum mechanics predicts the existence of global states that cannot be written as the product of states of local subsystems was first used in 1935 by Einstein, Podolsky and Rosen (EPR) to construct the famous EPR-paradox [EPR35]. Entanglement is the quantum feature behind the argument that quantum theory does not give a complete description of Nature, under the assumption that physical reality itself is described by ‘elements of reality’ (physical quantities whose values can be predicted with certainty without in any way disturbing the system) [EPR35]. It was however Shrödinger who first introduced the term ‘entanglement’\(^1\) and analyzed this property of quantum states [Sch35]. A quantifiable version of the EPR-paradox did not appear until 1964, in a noteworthy article by Bell [Bel64]. There the author proves that some predictions of quantum mechanics cannot be accurately reproduced by any local theory (for which the outcomes of measurements on a system are independent of operations on a separated system with which it might have interacted in the past). Bell’s definition of locality can be seen as the mathematical formulation of this same concept introduced by EPR [EPR35]. Considering locality as one of the assumptions of classical theory, Bell’s theorem implies that Nature is either nonlocal (hence nonclassical) or it allows instantaneous communication [Bel64].

The two distinct quantum features, entanglement and non-locality, are deeply related and both constitute valuable resources for quantum information processing tasks. However, the connection between them is still not completely understood and many important questions remain unopened. In this section, I present the theory related to each of the concepts, as well as their connection, which serves as the background for the results presented in the following chapters of this thesis. In section 2.1, I focus on entanglement concepts and some basic theory on operations on physical systems, which mainly support chapter 3. In section 2.3, I introduce the basics on non-locality together with its connection to quantum entanglement, its extension to multipartite case and more general nonlocality. This serves as support for chapters 4-7.

\(^1\)Actually, ‘Verschränkung’ is the original German term introduced by Schrödinger.
2.1 Quantum Entanglement

Here I define quantum entanglement and discuss how to detect it using entanglement witnesses and positive maps. I also relate both methods, which are equivalent from a theoretical point-of-view. Then I characterize these criteria in terms of optimality and classes of entanglement they are able to detect. Moreover, I explore the role of positive maps in describing physical operations, namely quantum channels and generalized measurements. Under this subject, I present the entanglement-breaking channels, which are useless for entanglement distribution. I also describe the structural physical approximation as a method to physically approximate non-physical maps.

2.1.1 Definition of Entanglement

Bipartite Scenario

Consider a pure quantum state $|\Psi\rangle$ of two subsystems, $A$ and $B$, defined on the composite Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. The quantum state $|\Psi\rangle$ is separable if it can be factorized as the product of states of the individual subsystems,

$$|\Psi\rangle = |\psi\rangle_A \otimes |\psi\rangle_B,$$

and is entangled whenever such decomposition is impossible. In the mixed-state case, a quantum state $\rho$ is separable when it can be written as the convex combination of product states [Wei89],

$$\rho = \sum_i p_i \rho_A^i \otimes \rho_B^i.$$  (2.2)

This definition carries a clear operational meaning: entangled states are those that cannot be obtained through local operations on product states — to prepare the different local states $\rho_A^i/B$ — and classical communication — that the parties, usually named Alice and Bob, use to classically correlate their preparations. According to this operational approach, it is natural to impose that the amount of entanglement cannot increase under LOCC operations.

Multipartite Scenario

In the multipartite scenario, characterizing a composite quantum state according to its separability properties is much more than classifying it as 'separable' or 'entangled'. But let us start with the most straightforward extension of bipartite separability (2.1) into the multipartite case. A quantum state of $N$ particles which can be factorized into local states

$$\rho = \sum_i p_i \rho_1^i \otimes \rho_2^i \otimes \ldots \otimes \rho_N^i,$$  (2.3)

is called completely separable. A state which cannot be represented as (2.3) certainly contains some entanglement. It is however relevant (in particular, from a resource point of view) to distinguish, for instance, the case where this entanglement is shared by pairs of particles or where it involves every particle of the system.
2.1. QUANTUM ENTANGLEMENT

For a complete characterization of multipartite entanglement one should then consider all possible groupings of particles of the multipartite system and study the entanglement between such groups. Following this idea, in [DC00] is introduced the concept of $k$-separability relative to a partition of the system. It evaluates the separability of the state for a specific partition and is defined as follows. Consider an $N$-partite quantum state $\rho$ and some splitting $S = \{s_1, \ldots, s_k\}$ of the corresponding quantum system into $k \leq N$ subsystems. The quantum state is called $k$-separable in the partition $S$ if it is completely separable according to that partition, that is, if it is separable according to (2.3) when the individual particles $i = \{1, \ldots, N\}$ are replaced by the groups of particles $i = \{s_1, \ldots, s_k\}$.

$$\rho = \sum_{i=1}^{k} p_i \tilde{\rho}_i^1 \otimes \tilde{\rho}_i^2 \otimes \ldots \otimes \tilde{\rho}_i^k. \quad (2.4)$$

Here $\tilde{\rho}_i^m$ represents the quantum state of the $m$th group of particles in the partition $S$. As mentioned, knowing the $k$-separability of each possible partition of the state would completely characterize its entanglement. However, it is clear that such a complete classification soon becomes too complex with the size of the system, and it is therefore convenient to consider less demanding, not complete, classifications that nevertheless provide useful information on the entanglement properties of a multipartite quantum state.

A possibility then is to classify the entanglement of a quantum state according to the maximum number of particles of the system which are entangled among themselves. A quantum state is said to contain at most $s$-entanglement if it can be written as a mixture of terms of the kind (2.4), where each group of particles has no more than $s$ elements [HHHH09]. If the $N$-partite state does not admit any sort of decomposition (2.4), that is, when all particles are simultaneously entangled, the state is called $N$-entangled or genuine multipartite entangled. It is not hard to see that a necessary and sufficient condition for the presence of genuine entanglement in a pure state is that it is entangled across every bipartition [HHHH09].

Now, given an arbitrary quantum state, the question is to decide whether it contains some entanglement. This is called the quantum separability problem, which is known to be NP-hard even in the simplest case of two-particle systems [Gur03]. Fortunately, for many relevant situations the problem is solvable, and an active line of research is to find new methods or classes of quantum states for which that occurs. In the following, I will consider only the bipartite scenario and introduce necessary and sufficient conditions for separability of any quantum state.

### 2.1.2 Entanglement Detection with Positive Maps and Entanglement Witnesses

The first necessary and sufficient conditions for the existence of entanglement were introduced by the Horodecki family [HHH96]. I start with the one that uses a particular kind of observables – the entanglement witnesses.

**Separability criterion with witnesses** [HHH96] – Consider a Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ and the space of operators $\mathcal{B}(\mathcal{H}_A) \otimes \mathcal{B}(\mathcal{H}_B)$ acting on it. A
CHAPTER 2. BACKGROUND

quantum state $\rho \in \mathcal{B}(\mathcal{H}_A) \otimes \mathcal{B}(\mathcal{H}_B)$ is entangled if and only if there exists a Hermitian operator $W = W^\dagger$ in $\mathcal{B}(\mathcal{H}_A) \otimes \mathcal{B}(\mathcal{H}_B)$, such that
\[
\text{tr}(W\rho) < 0 \quad \text{and} \quad \text{tr}(W\sigma) \geq 0,
\] for any separable state $\sigma \in \mathcal{B}(\mathcal{H}_A) \otimes \mathcal{B}(\mathcal{H}_B)$ [HHH96]. The operator $W$ is called an entanglement witness.

This condition arises from the geometric properties of sets of quantum states, defined in the space of Hermitian operators. Given that a convex sum of quantum states defines a new state, the set of quantum states $Q$ is closed and convex. The same applies to separable states, which then define a closed convex subset $S$ of $Q$. An entanglement witness defines an hyperplane on this Hermitian operator space, which separates it into two halves. One contains the set $S$, and therefore the expected value of $W$ in any state of this region is positive. On the contrary, quantum states from the other half-space provide a negative value to $\langle W \rangle$, and its entanglement is then detected. From this geometrical picture, it is clear that the negativity of $\text{tr}(W\rho)$ for a given $\rho$ is only a sufficient condition for $\rho$ to be entangled: the frontier between the set of entangled states and separable states is in general nonlinear and can only be defined by a continuous set of hyperplanes (see Fig. 2.1).

The other separability criterion is based on positive maps $\Lambda$: linear transformations from positive operators $A \geq 0$ to positive operators $\Lambda(A) \geq 0$. Before enunciating it, I introduce the concept of completely-positive (CP) maps, which define positive maps even when extended by the identity operator acting on a space of arbitrary dimension $k$, i.e.
\[
1_k \otimes \Lambda \geq 0.
\] (2.6)

As we will see below, CP maps always describe a physical operation.

**Separability criterion with positive maps [HHH196]** – A quantum state $\rho \in \mathcal{B}(\mathcal{H}_A) \otimes \mathcal{B}(\mathcal{H}_B)$ is entangled if and only if there is a positive, non-completely-positive map $\Lambda : \mathcal{B}(\mathcal{H}_B) \rightarrow \mathcal{B}(\mathcal{H}_A)$ such that
\[
\rho' = 1 \otimes \Lambda(\rho) < 0.
\] (2.7)

For any separable state $\sigma$, the action of a positive map on a subsystem always leads to a new separable state $\sigma' = 1 \otimes \Lambda(\sigma)$. On the contrary, if $\rho$ is entangled, the mapping is not physical since it can yield an operator $\rho'$ with negative eigenvalues. Again, considering a specific map $\Lambda$, the negativity of $\rho'$ is a sufficient but not necessary condition for the presence of entanglement, since it might happen that $\rho'$ is a positive operator. The first and most famous example of a positive not completely positive map used to detect entanglement is the transposition map, $\Lambda(A) = A^T$ [Per96b]. Partial transposition of a bipartite quantum state turns out to be both a simple and powerful method, as we will see in detail later on.

It fact, the previous criteria are completely equivalent and going from one formulation to the other is possible through the Jamiołkowski isomorphism, a bijective map between linear maps $\Lambda$ from the space of operators $\mathcal{B}(\mathcal{H}_A)$ to the space $\mathcal{B}(\mathcal{H}_B)$, and operators in $\mathcal{B}(\mathcal{H}_A) \otimes \mathcal{B}(\mathcal{H}_B)$. The isomorphism can be expressed as
\[
W = 1 \otimes \Lambda(\Phi_{d_A}^+) \quad \text{(2.8)}
\]
where $\Phi^+_d$ defines the projector onto the maximally entangled state in $\mathcal{H}_A \otimes \mathcal{H}_A$

$$|\Phi^+_d\rangle = \frac{1}{d}\sum_{i=1}^{d}|i\rangle.$$  \hfill (2.9)

or, in the opposite sense,

$$\Lambda_W(\rho) = d_A \text{tr}_A \left[ W(\rho^T \otimes 1) \right].$$  \hfill (2.10)

The Jamiolkowski isomorphism turns out to be very useful in quantum information science for its two-fold application: (i) a map $\Lambda$ is positive, not completely positive, if and only if $W$ is an entanglement witness, and (ii) $\Lambda$ is completely positive (and therefore, defines a quantum operation, as discussed later) if and only if $W$ is a quantum state.

An important remark is that the Jamiolkowski isomorphism does not imply that a given positive map $\Lambda_W$ and the corresponding entanglement witness $W_A$ are able to detect the same set of entangled states. In fact, positive maps provide a much stronger condition. Consider that the entanglement in some state $\rho$ is detected by both criteria. Then, following Eq. (2.10), the positive map condition detects entanglement in any state of the family $I \otimes A \rho I \otimes A^{-1}$, where $A$ is an invertible operation on subsystem B. One can easily see that the same does not necessarily hold for the entanglement witness condition. However, the use of positive maps to detect entanglement has a strong drawback: positive maps describe non-physical operations and the criterion requires previous state estimation process before it can be applied. On the contrary, entanglement witnesses are observables which can even be decomposed into local observables in an efficient way [GHB+02]. It is possible as well to optimize the complexity of these measurements with respect to, for instance, the number of measuring device settings [GHB+02].

### 2.1.3 Classification of Positive Maps and Entanglement Witnesses

Entanglement witnesses and positive maps can be classified according to the entangled states they are able to identify. The two basic categories are concerned with (i) the type of entanglement detected and (ii) the size of the set of identified entangled states. This classification is not only useful for the construction of appropriate entanglement detectors but also provides a good insight on the structure of the set of quantum states. In the following, I will interchange between the witnesses and maps framework, according to convenience. Mind though that both pictures are completely equivalent and all definitions can be translated from one language to the other by the use of the Jamiolkowski isomorphism (see (2.8) and (2.10)).

**Decomposable and Non-Decomposable Positive Maps/Witnesses**

Positive maps (and entanglement witnesses) can be divided into two classes: the *decomposable* class, composed by positive maps with a particularly simple representation in terms of completely positive operators (see below) and the *non-decomposable* class, which contains the remaining maps [Wor76]. This induces
the classification of the positive maps according to the kind of entanglement they are able to detect. Here, we consider two types of entanglement: one which is contained in quantum states with negative partial transposition (NPT states) and the other in states with positive partial transposition (PPT states) [Per96b].

**Decomposable Maps** – A positive map is said to be *decomposable* if it can be written as [Wor76]

\[
\Lambda = \Lambda_{CP}^1 + \Lambda_{CP}^2 \circ T,
\]

where \( \Lambda_{CP}^1 \) and \( \Lambda_{CP}^2 \) are CP maps and \( T \) is the transposition map. Decomposable maps are only able to detect NPT entangled states: if a state \( \rho \) is PPT then \( \mathbb{1} \otimes \Lambda(\rho) \) is positive and its entanglement will not be identified.

If we want to detect PPT-entangled states, we necessarily need to use the following more complex maps.

**Non-Decomposable Maps** – A positive map is *non-decomposable* when it cannot be written in the form (2.11). These are strongly connected to PPT entanglement: a positive map is non-decomposable if and only if it detects some PPT entangled state [LKCH00]. The characterization of non-decomposable maps is a hard mathematical problem and only specific examples have been found so far (see [Hal06, CK09] and references therein). This basically consists in the open problem concerning entanglement detection, as every NPT entangled state is already detected by the PPT criterion [Per96b].

**Optimal Entanglement Witnesses**

The notion of optimal entanglement witness was introduced in [LKCH00]. The underlying idea is that some entanglement witnesses are more useful than others, in the sense that they are able to identify a strictly larger set of entangled states. Consider \( D_W \) as the set of entangled states detected by witness \( W \). An entanglement witness \( W_2 \) is *finer* than the entanglement witness \( W_1 \) if \( D_{W_1} \subset D_{W_2} \), i.e. if \( W_2 \) detects all states detected by \( W_1 \), and some more. An optimal entanglement witness is such that there is no other witness which is finer [LKCH00], and corresponds to an hyperplane tangent to the set of separable states (see Fig.2.1). The next theorem enunciates an important necessary and sufficient condition for optimality.

**General condition for optimality** [LKCH00] – A witness \( W \) is optimal if and only if, for all positive operators \( P \) and real numbers \( \epsilon > 0 \), \( W' = W - \epsilon P \) is not an entanglement witness.

From a geometrical point of view, removing the positive component \( \epsilon P \) corresponds to performing modifications to the hyperplane defined by \( W \) until it becomes tangent to the set of separable quantum states \( S \). There, the witness is optimal and further changes would cause it to cross the set \( S \), making it negative for some separable states.

Although rather intuitive, this condition is in most cases hard to verify [LKCH00]. Nevertheless, in the case of decomposable maps, such definition of optimality identifies a single optimal map: the transposition map

\[
\Lambda_{opt}^{dec} = T.
\] 

(2.12)

This can be obtained by considering the decomposable representation (2.11). Optimality imposes that the completely positive component is zero, which leaves us with

\[
\Lambda_{opt}^{dec} = \Lambda_{CP} \circ T.
\] 

(2.13)
2.1. QUANTUM ENTANGLEMENT

![Diagram of convex sets](image)

Figure 2.1: Representation of convex sets in the space of Hermitian operators. $S$ is the set of separable states, $Q$ is the set of quantum states and $W$ the set of witnesses. Entanglement witnesses can be represented in two forms. In one, they are hyperplanes that split the space of Hermitian operators in two halfspaces, one of them completely contains the set $S$. $W_{\text{opt}}$ is an entanglement witness finer than the witness $W$. Then, in this representation, $W_{\text{opt}}$ is optimal only if it is tangent to the set of separable states. On the other hand, an entanglement witness defines a point in the set $W$ of Hermitian operators positive on product states. Here, optimal witnesses always lie on the border of the set $W$.

But this is not sufficient for $\Lambda$ to be optimal. Indeed, every entangled state detected by $\Lambda_{\text{CP}} \circ T$ is also detected by $T$, while the reverse is not necessarily true: the action of a CP map on a negative operator can transform it into a positive one. Hence transposition is the strongest and optimal map.

For $2 \times 2$ and $2 \times 3$ systems it turns out that every positive map/witness is decomposable [Wor76]. Two important consequences of this fact are: (i) partial transposition becomes also a sufficient criterion for entanglement and (ii) there exist no PPT entangled states for such low dimensional systems. In the case of non-decomposable maps, the situation is much more complex and optimality is usually hard to check. This generic problem is treated in detail in [LKCH00]. Recently, in [Bre06, Hal06] it was introduced a provably optimal non-decomposable map, called the Breuer-Hall map.

Entanglement distillation and PPT-entanglement. At this point, I introduce the concept of entanglement distillation, as it turns out to be closely related to PPT entanglement. Entanglement distillation describes the process of extracting pure entanglement from several copies of quantum states containing noisy entanglement, by LOCC operations. It was proven that a quantum state is distillable only if it is NPT-entangled [HHH08], but it remains as an open question whether this is also sufficient.

A non-distillable state is called bound entangled and evidently all PPT states are also bound entangled. Since pure entanglement cannot be extracted from them, these quantum states are useless for many applications. In the case of $2 \times 2$ and $2 \times 3$ systems, there is no PPT-bound entanglement, although NPT-bound entangled states might eventually exist [HHHH09].
2.2 Physical Operations and Completely Positive Maps

A different use of maps in the context of quantum theory concerns the description of physical operations, which are completely positive (2.6) and contractive maps, i.e. $\text{tr}(\Lambda(\rho)) \leq 1$ if $\text{tr}(\rho) = 1$. Any physical process on a quantum state $\rho$ can be expressed according to the Kraus representation

$$\Lambda(\rho) = \sum_k E_k \rho E_k^\dagger$$

(2.14)

where $E_k$ are operators that satisfy $\sum_k E_k E_k^\dagger \leq \mathbb{1}$ and are named Kraus operators [Kra83]. A trace-preserving map, i.e. $\text{tr}(\Lambda(\rho)) = 1$, $\forall \rho$, corresponds to a deterministic process and describes the action of a quantum channel. On the other hand, a strictly contractive map, $\text{tr}(\Lambda(\rho)) < 1$, $\forall \rho$, represents a probabilistic process and naturally describes a particular outcome of some generalized measurement. The complete measurement is represented by a set of contractive maps $\{\Lambda_m\}$ such that $\sum_m \text{tr}(\Lambda_m(\rho)) = 1$, for any $\rho$.

An interesting question is how to realize a given quantum operation in the laboratory. It turns out these can always be implemented by performing an unitary evolution $U$ on the joint system composed by the quantum system $Q$ and an ancilla $E$, followed by a projective measurement $P$ on $E$ [NC00]

$$\Lambda(\rho) = \text{tr}_E(P U \rho \otimes \rho_E U^\dagger P)$$

(2.15)

where $\rho_E$ is some state of the ancilla. Notice that if the quantum operation is deterministic, then there is no need to perform the projective measurement. The unitary operator can be built directly from the Kraus representation (2.14) and the dimension of the ancilla coincides with the number of Kraus operators. It is also known that a physical operation on quantum system of dimension $d$, can be represented with at most $d^2$ Kraus operators. Therefore, the implementation of $\Lambda$ requires, in the worst case, a unitary evolution on a $d^3$-dimensional Hilbert space, followed by projection on a $d^2$-dim ancilla [NC00].

2.2.1 Entanglement-breaking maps

A quantum channel $\Lambda$ for which $\mathbb{1} \otimes \Lambda(\rho)$ transforms any state $\rho$ into a separable state $\sigma = \mathbb{1} \otimes \Lambda(\rho)$ is called entanglement-breaking [HSR03]. Clearly, these channels are useless for entanglement distribution. In Ref. [HSR03], the following equivalence was obtained:

1. The channel $\Lambda$ is entanglement-breaking;
2. The corresponding state $W_\Lambda$ is separable;
3. The channel can be represented in the Holevo form

$$\Lambda(\rho) = \sum_k p_k \rho_k = \sum_k \text{tr}(M_k \rho) \rho_k,$$

(2.16)

with some positive operators $M_k \geq 0$ defining a generalized measurement, $\sum_k M_k = \mathbb{1}$, and states $\rho_k$ determined only by $\Lambda$.  

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The last property above means that the action of an entanglement-breaking channel can always be replaced by a measurement and state-preparation protocol: Alice performs a measurement on the input state \( \rho \) and sends the outcome \( k \) to Bob, which prepares the pre-agreed state \( \rho_k \). Moreover, from the separable decomposition of the state

\[
W_{\Lambda} = \sum_k p_k |v_k\rangle\langle v_k| \otimes |w_k\rangle\langle w_k|
\]

with \( |v_k\rangle \in \mathcal{H}_A \) and \( |w_k\rangle \in \mathcal{H}_B \), one obtains the following explicit Holevo representation of \( \Lambda \) [HSR03]

\[
\Lambda(\rho) = \sum_k \text{tr} \left( d_{AP_k} |\bar{v}_k\rangle\langle v_k| \right) |w_k\rangle\langle w_k|, \quad (2.18)
\]

where the overbar denotes the complex conjugation. Since \( \Lambda \) is trace-preserving, the positive operators \( \{ M_k \} = \{ d_{AP_k} |\bar{v}_k\rangle\langle v_k| \} \) define the properly normalized measurement performed by Alice while, from his side, Bob prepares pure states \( |\omega_k\rangle \).

Notice that the notion of entanglement-breaking channels, and consequent results, is also valid in the case of completely-positive contracting maps. For those, the Holevo decomposition is still possible for entanglement-breaking maps with the positive operators \( M_k \) defining a partial measurement, \( \sum_k M_k < 1 \) [HSR03].

2.2.2 Structural Physical Approximations to Non-Physical Operations

We have seen that a map which describes a physical operation is necessarily linear and completely positive. This implies that operations as quantum cloning [WZ82] or universal NOT gate [BcvHW99] are physically impossible. However, it is possible to find approximations to the action of such non-physical maps [BcvH96, BcvHW99]. The notion of structural physical approximation (SPA) introduced in [Hor03] allows for physically approximating a large class of nonphysical maps. It defines the SPA map of a linear Hermitian map \( \Lambda : B(\mathcal{H}_A) \to B(\mathcal{H}_B) \) as the completely-positive contractive map

\[
\tilde{\Lambda}(\rho) = \delta(\rho)1\text{tr}(\rho) + \gamma \Lambda(\rho) \quad (2.19)
\]

where \( \delta(\rho) \geq 0 \) is a linear function and \( \gamma > 0 \) a positive parameter. Moreover, the optimal SPA map is defined as the one which (i) minimizes the amount of added noise, i.e. has minimum ratio \( \delta(\rho)/\gamma \) for any \( \rho \), and (ii) maximizes the probability of implementation \( \text{tr}(\tilde{\Lambda}(\rho_\Lambda)) \), where \( \rho_\Lambda \) corresponds to the most probable output of the channel \( \Lambda \). Then the optimal SPA map has a simple and intuitive formulation

\[
\hat{\Lambda}(\rho) = p\text{tr}(\rho) \frac{1}{d_B} + (1 - p)\Lambda(\rho) \quad (2.20)
\]

where \( 0 < p < 1 \) is the minimum amount of depolarized noise that can be added to \( \Lambda \) such that it becomes physically implementable. Notice that the SPA keeps the structure of the output of \( \Lambda \): the generalized Bloch vector of \( \hat{\Lambda}(\rho) \) is simply the re-scaling of the vector for \( \Lambda(\rho) \) by a factor \( 1 - p \).


2.3 Quantum Nonlocality

2.3.1 Nonlocality, Bell Theorem and Bell inequalities

The simplest framework to study nonlocality starts with two distant observers which perform local measurements on a shared physical system \( \rho \). The parties, Alice and Bob, can freely choose their measuring settings, labeled \( x \) and \( y \), and obtain outcomes \( a \) and \( b \), respectively. The number of measuring settings and outcomes is, in principle, arbitrary. It is required, however, that (i) each local measurements define space-like separated events, and (ii) the choice of the measuring setting is made by each individual observer at the moment of measurement. Then, the formulation of local theory introduced in [Bel64] assumes that the joint probability of observing outcomes \( a \) and \( b \), given the choice of inputs \( x \) and \( y \), is described by a hidden (classical) variable \( \lambda \) and local probability distributions \( P^A \) and \( P^B \) according to

\[
P_L(ab|xy, \rho) = \int d\lambda \omega(\lambda) P^A(a|x, \lambda) P^B(b|y, \lambda), \tag{2.21}
\]

where \( \omega \) represents the probability measure according to which \( \lambda \) is distributed [Bel64]. The local probability distributions can only depend on the measurement choice and on the hidden-variable \( \lambda \). This shared variable is then the only responsible for any correlations that might exist among the outcomes: according to the locality assumption a system cannot, in any form, be influenced by space-like events. The variable \( \lambda \) can be chosen as general as a continuous multidimensional parameter, and be interpreted as the result of the interaction of both subsystems at some moment in the past.

Any theory which is not described by a local model (2.21) is called nonlocal. Mind that the formulation of nonlocality is completely independent from the specific description of the state and the measurement procedure in a given physical theory; it merely invokes the possibility of performing experiments where the observers can choose their measurements and obtain outcomes. Remarkably, Bell’s theorem proves that quantum theory is nonlocal, that is, the outcome probabilities

\[
P_Q(ab|xy, \rho) = \text{tr}(\rho M^a_x \otimes M^b_y) \tag{2.22}
\]

where \( \sum_a M^a_x = 1 \), \( M^a_x = (M^a_x)^\dagger \) define some generalized measurement on Alice’s subsystem (and analogous for Bob), cannot be modeled by any distribution \( P_L \) (2.21) [Bel64]. For that, Bell considered the correlations arising from measurements on a composite system of two spin-1/2 particles in the (maximally-entangled) singlet state

\[
|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), \tag{2.23}
\]

where \( |0\rangle \) and \( |1\rangle \) respectively represent the up and down spin state. In addition, in [Bel64] it is also shown that the outcomes of measurements on a local system can always be explained by a local hidden-variable model

\[
P_Q(a|x) = P_L(a|x) = \int d\lambda \omega(\lambda) P^A(a|x, \lambda), \tag{2.24}
\]

and therefore it is necessary to consider composite systems in order to rule out the existence of hidden-variables.
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Inspired by Bell's theorem, in 1969, Clauser, Horne, Shimony and Holt introduce the CHSH inequality, a key element in the first experimental proposal to test the local character of Nature [CHSH69]. Consider an experiment where two parties measure two local observables, \( \{A_1, A_2\} \) and \( \{B_1, B_2\} \), with possible outcomes \( \pm 1 \). The following algebraic combination of expected values of joined measurements

\[
\langle A_1 B_1 \rangle + \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle - \langle A_2 B_2 \rangle | \leq 2 \tag{2.25}
\]

is bounded by 2 for any local theory (2.21). Equation (2.25) defines the CHSH-inequality and its violation immediately indicates the presence of nonlocal correlations. Quantum mechanics offers a violation up to the maximum value of \( 2\sqrt{2} \), known as Tsirelson's bound [Ts180], which can be attained by measurements on a singlet state (2.23). (See Appendix A for a more detailed characterization of this inequality.)

In general, inequalities that distinguish local from nonlocal correlations are called Bell inequalities, and they can be defined for any number of parties, settings or outcomes. The violation of a Bell inequality consists on the most common method to identify nonlocal correlations arising from local measurements on a quantum state. This provides a sufficient (but not necessary) criteria for nonlocality, and for that reason each Bell inequality is suited to detect some set of nonlocal correlations. In fact, Bell inequalities act in very similar way to entanglement witnesses. It follows from the fact that local (classical) probability distributions (2.21) define a convex set \( C \) in the space of probabilities. A Bell inequality is then represented by an hyperplane which splits this space into halfspaces, one containing the entire set \( C \). Consequently, to fully describe the set of local theories, it is necessary to consider a complete set of Bell inequalities, which will be finite since \( C \) is a polytope \(^2\). The minimal complete set of such Bell inequalities is composed by facets of the local polytope, which are hyperplanes tangent to the border of \( C \), in a subspace of maximal dimension. These facet Bell inequalities are the analog of the optimal entanglement witnesses described previously. (See Appendix A for a detailed characterization of sets of local and quantum probability distributions.)

An important remark on Bell inequalities is the link between two intrinsic quantum features, nonlocality and non-commutativity of observables. Any (bipartite) Bell inequality must always consider, at least, two non-commuting observables per party in order to identify nonlocal correlations. This follows directly from Fine's result [Fin82]: the existence of a local model for the correlations of \( P(ab|xy) \), obtained by measuring some discrete set of local observables \( \{O_x, O_y\}_{x=1...m, y=1...n} \), is equivalent to the existence of a well-defined joint probability distribution for all observables

\[
P(a_1 \ldots a_m b_1 \ldots b_n | 1 \ldots n, 1 \ldots m), \tag{2.26}
\]

which returns \( P(ab|xy) \) as marginal distributions.

To finish, notice that although local theories are confined to polytopes in a probability space (which implies that, after systematically listing all the facet

\(^2\)The first Bell experiment with reliable results was only accomplished in 1981 by Aspect, Grangier and Roger [AGR81]. Their results were compatible with quantum predictions and, up to loopholes, proved that Nature is nonlocal.

\(^3\)See appendix A
Bell inequalities, it is possible to know whether a given distribution is local or not), it is hard in general to decide if measurements on a given quantum state can produce nonlocal correlations. The first reason is that describing the local polytope soon becomes a too complicated mathematical problem with the dimension of the probability space (which easily occurs when increasing the number of settings or outcomes). Second, even if a quantum state is local for Bell experiments with a given number of settings and outcomes, increasing these values might reveal its nonlocality; at the end, locality in the standard scenario can only be assured by the construction of a local model (2.21) for arbitrary local measurements on the state.

2.3.2 Relation between Entanglement and Nonlocality

What is the connection between entanglement and nonlocality? Their definition invokes completely different concepts: while entanglement is based on the Hilbert space structure of quantum theory, and the possibility of factorizing composite states into local components (2.2), nonlocality only makes considerations on the nature of the observed outcome correlations of measurements on such composite systems. However, it is easy to identify a first link: any separable state is local, in the sense that the outcomes of local measurements on it can always be described by a local model (2.21). In fact, if a state $\rho$ has a separable representation (2.2), the properties of the trace imply

$$P_Q(ab|xy) = \text{tr}(\rho M_a^x \otimes M_b^y) = \sum_i p_i \text{tr}(\rho_i^A M_a^x) \text{tr}(\rho_i^B M_b^y)$$

(2.27)

which can always be described by a local hidden-variable model according to observation (2.24) [Wer89].

In the other direction, Gisin’s theorem shows that any pure bipartite entangled state violates a Bell inequality, hence is nonlocal [Gis91]. Surprisingly, this is not true for mixed states: in 1989, Werner constructed a local model for projective measurements on an entangled mixed quantum state which belongs to a family of states that became known as Werner states [Wer89]. Such model strongly relies on the symmetry properties of this family, which I now briefly summarize.

Definition of Werner states

Werner states constitute a family of bipartite quantum states defined on a $d \times d$ Hilbert space, characterized by being invariant under local unitary transformations $U \otimes U$. That provides them the following integral representation

$$\rho_W = \int dU(U \otimes U)\rho(U^\dagger \otimes U^\dagger)$$

(2.28)

where $U$ are unitary operators $UU^\dagger = U^\dagger U = 1$, $\rho$ represents a quantum state and $dU$ denotes Haar measure over the unitary group [Wer89]. These states are completely described by a single parameter $\phi = \text{tr}(F \rho_W) = \text{tr}(F \rho)$,

$$\rho_W = \frac{1}{d(d^2 - 1)} [(d\phi - 1)F + (d - \phi)1] .$$

(2.29)
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Here $F$ is the flip operator, $F|\psi\rangle|\varphi\rangle = |\varphi\rangle|\psi\rangle$, and $-1 \leq \phi \leq 1$ is an entanglement measure of $\rho_W$: the state is entangled for $\phi \geq 0$ [Wer89]. Notice that $\phi = 1$ corresponds to a projector on the symmetric subspace of $H$ while $\phi = -1$ is a projector on the anti-symmetric subspace.

**Werner’s local model (projective measurements)**

In general, constructing a local hidden-variable model for a given quantum state represents a complicated task, since one must guarantee that the local distribution (2.21) reproduces the quantum statistics (2.22) for any choice of local measurement operators. We will now see how the choice of $UU$-invariant states allows a considerable simplification of the problem. In the case of Werner states, outcomes of joint local measurements follow the probability distribution

$$P_Q(ab|xy, \rho_W) = \frac{1}{d(d^2 - 1)} \left[ (d - \phi)\text{tr}(M_x^a)\text{tr}(M_y^b) + (d\phi - 1)\text{tr}(M_x^aM_y^b) \right].$$

In his model, Werner restricts the measurements to be projective (i.e., $\sum_a \Pi_a^x = \mathbb{1}$ such that $\Pi_a^x\Pi_a^x = \delta_a \Pi_a^x$ [Wer89]). Then, it is sufficient to consider one-dimensional projectors (which implies $\text{tr}(\Pi_a^x) = 1$), since the local response functions in (2.21) for projections into higher dimensional spaces can always be obtained by the sum of one-dimensional projections. Thus, the distribution of outcomes (2.30) simplifies into

$$P_Q(ab|xy, \rho_W) = \frac{1}{d(d^2 - 1)} \left[ (d - \phi) + (d\phi - 1)\text{tr}(\Pi_a^x\Pi_b^y) \right].$$

The local model that reproduces such distribution takes the hidden-variable $\lambda$ to be a unit vector in $\mathbb{C}^d$, distributed according to the measure induced by the Haar measure $\omega$ (which is the only invariant under unitary operations). The intuition behind this choice is that $\lambda$ can be seen as the classical analog of the shared quantum state and therefore should capture most of its properties. Again, the reasoning applied to the outcome distributions leads to consider particular local response functions. They are chosen such that unitary transformations $U$ on $\lambda$ (the analog of the quantum state) are equivalent to $U^\dagger$ transformations on the measurement direction (the analog of the measurement operators),

$$P(a|x, U\lambda) = P(a|U^\dagger x, \lambda).$$

Here, using some abuse of notation, $x$ represents the measurement vector itself. Given this, the most natural choice is the “quantum-like” response function

$$P_A(a|x, \lambda) = \langle \lambda|\Pi_a^x|\lambda \rangle$$

which is used by Alice. This constrains Bob’s response function to be associated to a unique positive operator

$$\hat{P}_B(b|y) = \int d\lambda \omega(\lambda)\text{tr}(\Pi_B^b\langle \lambda|\lambda \rangle).$$

for which the local probabilities are given by a quantum-like expression

$$P_L(ab|xy, \rho_W) = \text{tr}(\hat{P}_B(b|y)\Pi_a^x).$$
Since we want this distribution to match the quantum prediction (2.31), the operator $\hat{P}_B(b|y)$ must be of the form

$$\hat{P}_B(b|y) = \frac{1}{d(d^2 - 1)} [(d - \phi)\mathbb{1} + (d\phi - 1)\Pi^b_y].$$ \hspace{1cm} (2.36)

Now, it remains to find the local response function associated to the operator (2.34), which follows (2.32) and is valid for a parameter $\phi$ as small as possible (we want to maximize the overlap between the entanglement and the local regions). An important remark is that it is sufficient to consider the simplest case where $\Pi^a_x = \Pi^b_y$, since by (2.36) it is already guaranteed that the local model reproduces the quantum prediction. This suggests the following response function

$$P_B(b|y, \lambda) = \begin{cases} 1 & \text{if } \langle \lambda | \Pi^b_y | \lambda \rangle = \min_i \langle \lambda | \Pi^i_y | \lambda \rangle \\ 0 & \text{otherwise} \end{cases},$$ \hspace{1cm} (2.37)

that, interestingly, captures the anti-correlation properties of Werner states. This is particularly simple to see in $d = 2$, where they are the admixture of the singlet (2.23) with white noise

$$\rho_W = p|\psi^-\rangle \langle \psi^-| + (1-p)\frac{\mathbb{1}}{4}$$ \hspace{1cm} (2.38)

for $p = (1 - 2\phi)/3$. Computing the integral defined by (2.35) for the chosen response functions for Alice (2.33) and Bob (2.37), we finally obtain that the local model is valid for

$$\phi = -1 + \frac{d + 1}{d^2},$$ \hspace{1cm} (2.39)

that corresponds to a simultaneously entangled and local Werner state. Consequently, any quantum states in the range

$$-1 + \frac{d + 1}{d^2} \leq \phi < 0$$ \hspace{1cm} (2.40)

are local and entangled as well, since they correspond to the mixture of the state with (2.39) and maximally mixed state (which obviously admits a local model).

Werner’s result proves that in the case of mixed quantum states, entanglement does not imply nonlocality, when observers are restricted to projective measurements. Would this still hold if one allows generalized local measurements to be performed? The affirmative answer can be found in Barrett’s local model for general measurements on Werner states, introduced as follows [Bar02].

**Barrett’s local model (generalized measurements)**

Barrett’s local model extends the results from Werner [Wen89] and reproduces the correlations arising from generalized measurements on a range of Werner states [Bar02]. One should now consider the general probability distribution (2.30) (instead of the simplified expression for projective measurements (2.31)). Barrett starts then by considering generalized measurements described by operators proportional to one-dimensional projectors $\Pi^a_x$

$$M^a_x = c^a_x \Pi^a_x$$ \hspace{1cm} (2.41)
2.3. QUANTUM NONLOCALITY

where \( 0 \leq c^a_x \leq 1 \) and \( \sum_a c^a_x = 1 \). Under this restriction, the quantum statistics is given again by the expression (2.31), but affected by weights \( c^a_x \) and \( c^b_y \):

\[
P_{Q}(ab|xy, \rho_W) = \frac{1}{d(d^2-1)} c^a_x c^b_y \left[ (d - \phi) + (d\phi - 1)\text{tr}(\Pi_x^a \Pi_y^b) \right]. \tag{2.42}
\]

For certain \( \phi \), this distribution is obtained by the following local model. Take the hidden-variable \( \lambda \) to be the same as in Werner’s model, also with the distribution induced by the Haar measure on unitaries. Alice’s local distribution is given by

\[
P(a|x, \lambda) = \text{tr}(|\lambda\rangle\langle\lambda| M^a_x) \Theta(\text{tr}(|\lambda\rangle\langle\lambda| \Pi^a_x) - \frac{1}{d}) + \text{tr}(\frac{1}{d} M^a_x)(1 - \sum_k \text{tr}(|\lambda\rangle\langle\lambda| M^a_k) \Theta(\text{tr}(|\lambda\rangle\langle\lambda| \Pi^a_k) - \frac{1}{d})). \tag{2.43}
\]

where \( \Theta(x) \) is the Heaviside step function. The response function (2.43) distinguishes between two scenarios:

1. \( \text{tr}(|\lambda\rangle\langle\lambda| \Pi^a_x) \leq 1/d \) - If the projection of the vector \( |\lambda\rangle \) in the measurement vector is lower or equal to the average value \( 1/d \), the local model assumes that Alice and Bob are sharing noise. Then, the probability of the outcome \( a \) is given by the product of the quantum-like probability of \( a \) in presence of white noise \( \text{tr}(M^a_x \mathbb{1}/d) = c^a_x/d \) with the quantum-like probability of obtaining any outcome \( k \) which also verifies \( \text{tr}(|\lambda\rangle\langle\lambda| \Pi^a_k) \leq 1/d \);

2. \( \text{tr}(|\lambda\rangle\langle\lambda| \Pi^a_x) > 1/d \) - In the other case, the model assumes that Alice and Bob share correlations and the probability of \( a \) is increased by the quantum-like factor \( \text{tr}(|\lambda\rangle\langle\lambda| M^a_x) \).

Bob also uses a local strategy strongly inspired by the quantum

\[
P(b|y, \lambda) = \frac{c^b_y}{d-1} (1 - \text{tr}(|\lambda\rangle\langle\lambda| \Pi^a_y)) \tag{2.44}
\]

but taking into consideration that his result should be anti-correlated with Alice’s.

Such local model is valid for

\[
\phi = \frac{d^d - (d-1)d^{-1}(3d-1)}{d^{d+1}} \tag{2.45}
\]

which is a negative quantity and therefore corresponds to an entangled Werner state. By the same reasoning as in Werner’s model, this induces a local model for all entangled states in the interval

\[
\frac{d^d - (d-1)d^{-1}(3d-1)}{d^{d+1}} \leq \phi < 0. \tag{2.46}
\]

One can easily see that this also holds for any generalized measurement. From the spectral theorem, every measurement operator element which describes a given measurement has eigendecomposition

\[
M^a_x = \sum_k c^a_{x}^{a k} \Pi^a_k, \tag{2.47}
\]
with real eigenvalues $0 \leq c^a_k \leq 1$. Since the previous derivation holds for $M^a_k = c^a_k \Pi^a_k$, we can consider that each observer performs a “fine-grained” measurement and outputs $a$ whenever he obtains the result $a_k$ [Bar02].

Comparing the regions for which the local models for projective and generalized are valid, we observe that it is smaller for the last. Although this coincides to what one might expect — generalized measurements are able to better reveal the nonlocality of states — it is not possible to extract such conclusion from these results. If the local models are not optimal, the obtained local-entangled regions cannot be considered more than a subset of the real ones. In fact, it is known that in the case of two-qubits, Werner’s local model is not tight [AGT06]. Nevertheless, we are able to conclude that at least in the region where Barrett’s model is valid (2.46), Werner states are entangled but unable to violate any Bell inequality for any kind of local measurements.

We are now faced with a fundamental question: is it true that there exist entangled quantum states which do not contain any sort of nonlocality? One might argue that the standard scenario (single local measurements on a single copy of the state) is not the most general one. Several generalizations to the concept of nonlocality, or more generic frameworks where to test it, were introduced to answer such criticism. Finding methods to reveal the nonlocality of entangled quantum states which are local in Bell’s original scenario is generally known as revealing hidden-nonlocality.

The first generalization of nonlocality is to consider that, instead of a single measurement, parties are allowed to perform a sequence of measurements [Pop95]. As we will see, this is equivalent to another extended scenario where parties can make local filtering, i.e., perform the Bell test only if their local states are subject to some local interaction [Gis96]. Other possibility is that Alice and Bob perform collective local measurements on several copies of the shared state, and eventually combine this with sequence of measurements [Per96a]. Finally, a more extended framework allows the parties to add any local resources (as shared local quantum states) which combined with their initially shared quantum state would manifest some nonlocality [MLD08]. In the following, I will provide a description of these new scenarios and the most important related results.

**Sequential measurements or local filtering**

The concept of hidden-nonlocality introduced in [Pop95] studies the outcome correlations when observers are allowed to perform sequential measurements on a single copy of the state. Following this procedure, Popescu proved that entangled Werner states with local dimension $d \geq 5$, which admitted a local model for projective measurements, violate the CHSH inequality after a suitable projection [Pop95]. Notice that such violation only occurs for a certain outcome of the first measurement, but this restriction does not affect the full generality of the results since the choice of settings for the CHSH inequality is only performed after the first measurement. Intuitively, this means that the hidden-variable does not know in advance which measurement setting will be used, and therefore cannot “avoid” undesirable tests by providing outputs for the first measurement which will always be discarded.\footnote{This is, in fact, the essence of the detection loophole. If particles are able to escape detection when they see that the choice of measurement is inconvenient, it is even possible to...} This observation can be rigorously justified by...
2.3. QUANTUM NONLOCALITY

the detailed analysis of [TBD+97] and [idZHHH98], which goes as follows.

The goal then is to study the distribution of outcomes for sequences of local measurements on a bipartite quantum system $\rho$. Consider the simplest case, where parties are allowed to perform two successive measurements: the generalization for $N$ measurements is straightforward. Following (2.21), a local model for the joint probability distribution of outcomes reads

$$P_L(a_1a_2b_1b_2|x_1x_2y_1y_2, \rho) = \int d\omega(\lambda)P^A(a_1a_2|x_1x_2, \lambda)P^B(b_1b_2|y_1y_2, \lambda).$$

(2.48)

Suppose that measurements are performed in the time sequence $t_2 - t_1 > 0$, but assuring that Alice and Bob’s events are spacelike separated. Since the choice of measurement setting for the second measurement is performed after the first outcome was obtained, causality imposes that the first measurement does not depend on the second,

$$P(a_1a_2|x_1x_2, \lambda) = P(a_2|x_2, a_1x_1)P(a_1|x_1, \lambda),$$

(2.49)

and similar in Bob’s side. Introducing the extra condition (2.49) in the definition of local model (2.48), we obtain the definition of local model for sequential measurements or local causal model [TBD+97, idZHHH98]:

$$P_L(a_1a_2b_1b_2|x_1x_2y_1y_2, \rho) = \int d\omega(\lambda)P(a_2|x_2, a_1x_1)P(a_1|x_1, \lambda)P(b_2|y_2, b_1y_1, \lambda)P(b_1|y_1, \lambda).$$

(2.50)

In the case of $N$ sequential measurements, one generalizes (2.49) and imposes that every measurement step can only depend on the past events. This defines a different model than the usual one (2.48), hence correlations that cannot be written as (2.50) are called hidden-nonlocal. Notice that if measurements are not spacelike separated, the independence condition (2.49) is not valid, which would allow local models to “mimic” nonlocal correlations. It is not hard to see that hidden-locality is stronger than locality in the sense a hidden-local model (2.50) implies a local model. Moreover, although the local causal model seems a particular case of the local model this is not necessarily true: even if correlations are hidden-local this only implies that they are local if $P_Q(a_1b_1|x_1y_1, x_2y_2) = P_Q(a_1b_1|x_1y_1)$.

Now, as noted in [idZHHH98], the violation of a Bell inequality for a specific quantum state arising from the sequence of measurements, i.e. for a single outcome of that process, is sufficient to conclude that the outcome distribution is not locally-causal. It follows directly from the fact that (2.50) induces a local model (2.21) for every post-measurement state

$$\rho_{a_1b_1} = \frac{M_{a_1b_1, \rho}M_{a_1b_1, 1^1}}{\text{tr}(M_{a_1b_1, \rho}^1M_{a_1b_1, 1^1})}$$

(2.51)

with local measurement operators $M_{a_1b_1} = M_{a_1} \otimes M_{b_1}, \sum_{a,b_1} M_{a_1b_1}^1 M_{a_1b_1}^1 = 1$. This confirms the reasoning in [Pop95], and shows that Werner states with $d \geq 5$ have their hidden-nonlocality revealed by a suitable projection of the local obtain violations of the CHSH-inequality above the quantum bound [Gis96].
subspaces. Consequently, the framework of sequential measurements is completely equivalent to the scenario of local filtering introduced by Gisin [Gis96]. There the parties can subject their particles to the interaction with some local environment, which will modify their shared state (which becomes filtered). Only after this process, they will choose the measurement setting and perform the CHSH test.

Although sequential measurements/local filtering consist of a powerful tool to study the nonlocal properties of outcome correlations, it is possible to consider a more general framework, for which the parties are also allowed to perform collective measurements on several copies of the quantum state.

Collective measurements

Peres was the first to consider a scenario for nonlocality where collective tests together with successive measurements are allowed [Per96a]. Using a measurement scheme inspired by distillation protocols, he proves that it is possible to reveal the hidden-nonlocality of $d = 2$ Werner states. The example takes five copies of two spin-1/2 particles in the state (2.38) with noise parameter $p = 1/2$ (for which Werner’s model is valid) and shows that, for some outcome of local collective measurements, the obtained quantum state violates the CHSH inequality. Hence the initial state Werner state contains hidden-nonlocality, although it admits a local model for projective measurements on single copies. The used derivation did not give the possibility of testing the asymptotic regime, where the number of copies $N$ tends to infinity. This process would however reveal the hidden-nonlocality of Werner states for the range where they are distillable. Based on this link between hidden-nonlocality and entanglement distillability, Peres conjectured that bound-entangled states are unable to violate any Bell inequality [Per99]. The Peres’ conjecture remains remarkably resistant to all attempts to disprove it [HHHH09].

Extra local resources

Finally, a further extension of nonlocality is to allow parties to share extra local resources, which would serve as “catalysts” of nonlocality. A similar, but less demanding, framework is considered in [MLD08]. There, the catalysts can be nonlocal but are still restricted, as we will see. The results of [MLD08] provide, nevertheless, an interesting link between bipartite entanglement and hidden-nonlocality. They consider that the parties share an extra state $\rho$, which does not violate the CHSH inequality after successive measurements, and the goal is to reveal the nonlocality of some quantum state $\sigma$. It is proven that $\sigma$ is entangled if and only if there is a state $\rho$, not violating a CHSH inequality after sequential measurements on a single copy, such that $\rho \otimes \sigma$ does. If $\sigma$ was separable, it could never do the task, so the claim is that there must be some hidden-nonlocality in any entangled state $\sigma$ [MLD08]. Although this constitutes a quite interesting result, it would become stronger if $\rho$ did not violate the CHSH inequality after sequential measurements on $N$ copies of the state, or even more if $\rho$ was local at the single-copy level.

To conclude, these results seem to indicate that every entangled state contains some hidden-nonlocality, which is defined by the failure of much more general local tests than the one originally introduced by Bell (2.21). However,
not much is known about the activation of nonlocality, that is, the case where sequential measurements are not allowed. The closest result is possibly that collective measurements can improve the violation of some Bell inequalities [LD06]. So, do all entangled quantum states violate some Bell inequality, when local collective measurements are allowed? Is Peres’ conjecture true?

2.3.3 Quantum nonlocality in the EPR-2 approach

Bell’s theorem is of statistical nature: it proves that outcome probabilities predicted by quantum theory are incompatible with a local model (2.21). Therefore, a Bell test consists of performing local measurements on a given shared quantum state, repeating this procedure as many times as necessary to obtain representative statistics. In [EPR92], the authors asked the question whether it would be possible for the ensemble of particles used to perform the Bell test to be composed of two fractions: one composed of particles having a local behavior, i.e., with statistics obeying (2.21), and the other with particles that followed some nonlocal model, not necessarily the quantum one. No Bell inequality can distinguish this scenario from the standard one, where all the particles have quantum nature. It follows from the fact that it is always possible to decompose the quantum probability distribution on a convex sum of local, $P_L$, and nonlocal, $P_{NL}$, components

$$P_Q(ab|xy) = p_L P_L(ab|xy) + (1 - p_L) P_{NL}(ab|xy). \quad (2.52)$$

The quantum distribution arises when local measurements $x$ and $y$, yielding outcomes $a$ and $b$, are performed on some quantum state. Since the decomposition (2.52) does not depend on the measurement settings nor on the outcomes, the weight $0 \leq p_L \leq 1$ is defined as an intrinsic property of the quantum state. Also, it only makes sense to define it as the maximum value such that (2.52) holds for any $(a,b|x,y)$. Without this restriction, the above decomposition would be trivial. It follows from the fact that the local distribution (2.21) is a particular case of nonlocal distribution,

$$P_{NL}(ab|xy) = \int d\lambda \omega(\lambda) P(ab|xy,\lambda), \quad (2.53)$$

since putting $p_L = 0$ would always satisfy (2.52). The decomposition (2.52) is usually called EPR2-decomposition after the authors and for obvious reasons.

Now, a new question on quantum nonlocality arises: is quantum theory incompatible with a local model for every pair of particles in the ensemble? Or, are there examples of states for which the local weight $p_L$ is zero? In [EPR92] the authors provide an affirmative answer to this question by showing that the singlet state (2.23) is fully nonlocal, in the sense that $p_L = 0$. Moreover, it is proven that for partially entangled states

$$|\psi\rangle = \cos \theta |0\rangle + \sin \theta |1\rangle \quad (2.54)$$

the local weight is strictly positive, $p_L > 0$. Such result was obtained after the construction of a local model for which the EPR-2 decomposition (2.52) reproduces quantum statistics for any parameter $\theta$. This requires finding suitable parameters $p_L$ and checking whether the nonlocal probability distribution
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is well-defined. Note that these local models are more restrictive than those by Werner or Barrett \(^5\), essentially because now the local model, combined with some general nonlocal distribution, should reproduce a nonlocal quantum distribution. Non-trivial bounds on \(p_L\) have been obtained for partially-entangled pure two-qubit and three qubit states [Sca08, BGS10] and for some mixed states [ZRS09].

In [BKP06], it is proven that any bipartite \(d \times d\) maximally-entangled quantum state (2.9) is fully nonlocal, \(p_L = 0\). Their method allows to identify fully nonlocal states without the need for the technical assumptions of [EPR92]. It goes as follows. Consider a Bell inequality

\[
I = \sum_{abxy} c_{abxy} P(ab|xy) \tag{2.55}
\]

with real weights \(c_{abxy}\). Using the EPR-2 decomposition (2.52), write the value obtained after performing chosen measurements on a quantum state \(\psi\) as

\[
I_Q = p_L I_L + (1 - p_L) I_{NL}. \tag{2.56}
\]

According to (2.56), the local fraction of \(\psi\) is given by

\[
p = \frac{I_{NL} - I_Q}{I_{NL} - I_L} \tag{2.57}
\]

where we consider \(I_{NL} \geq I_Q > I_L\). If the quantum violation \(I_Q\) coincides with the nonlocal bound \(I_{NL}^* = \max I_{NL}\) of the inequality, the state \(\psi\) is fully nonlocal. This is accomplished in [BKP06], by the construction of a chained Bell inequality, with \(N\) settings and \(d\) outcomes, for which there exist measurements on maximally entangled states that asymptotically reach the nonlocal bound when \(N \to \infty\). Hence maximally-entangled states \(\Phi^+_d\) are fully nonlocal.

In the next section, I show how full nonlocality is associated to a nonlocal phenomena called GHZ-paradox. This is will prove that states with \(p_L = 0\) exist for any number of parties. But, before that, I will introduce the main results on nonlocal phenomena which can be found on the multipartite scenario.

2.3.4 Multipartite Nonlocality

The study of quantum nonlocality in multipartite systems is still in a primitive stage, and even the generalization of most results from the bipartite scenario has not yet been accomplished. Before providing an overview on relevant results concerning multipartite nonlocality, lets start by defining a local model in this new scenario. In fact, the extension of nonlocality to multipartite systems follows a parallel process to the one for multipartite entanglement \(^6\). With \(N\) parties, the structure of correlations becomes much more complex, and it is therefore not enough to characterize a distribution simply as local or nonlocal. Lets start however by the straightforward extension of bipartite local model (2.21) to a system of \(N\) separated parties

\[
P_L(a_1a_2\ldots a_N|x_1x_2\ldots x_N, \rho) = \int d\lambda \omega(\lambda) P_1(a_1|x_1, \lambda)P_2(a_2|x_2, \lambda)\ldots P_N(a_N|x_N, \lambda), \tag{2.58}
\]

\(^5\)See section 2.3.2.

\(^6\)See section 2.1.1.
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which defines a completely local model. Here, all parties of the system are compatible with classical theory since they are only allowed to share local correlations. Most available Bell inequalities for the multipartite scenario test if a given outcome distribution can be described by these completely local models (2.58) [GBP98, Mer90a, idZB02, WW01a]. But then, they do not provide any information on the truly multipartite nonlocal content of the distribution: a violation of these inequalities could either be a result of two-party nonlocality or genuine N-partite. Identification of truly multipartite nonlocality is only possible considering the following generalization of bipartite Bell inequalities.

Svetlichny-like Inequalities and Genuine Multipartite Nonlocality

In 1987, Svetlichny constructed inequalities for three parties that are able to identify genuine nonlocal correlations, which correspond to nonlocal correlations that are established between all the particles of the system [Sve87]. For that, it is necessary to introduce the concept of hybrid local-nonlocal models,

\[
P_{A_1:A_2:A_3}(a_1a_2a_3|x_1x_2x_3) = \int d\lambda \omega(\lambda) P(a_1|x_1, \lambda) P(a_2a_3|x_2x_3, \lambda),
\]

(2.59)

Here, party \( A_1 \) is classically correlated to parties \( A_2 \) and \( A_3 \), while these two are allowed to share any nonlocal correlations. A given probability distribution \( P \) violates a Svetlichny inequality if it cannot be written as the convex combination of these hybrid models (2.59), i.e.,

\[
P \neq p_1P_{A_1:A_2:A_3} + p_2P_{A_2:A_1:A_3} + p_3P_{A_3:A_1:A_2}
\]

(2.60)

with \( p_1 + p_2 + p_3 = 1 \). Notice that there is no need to explicitly consider the completely local distribution (2.58), since it is already included in any of the hybrid models (2.59). The distribution \( P \) is then called genuine tripartite nonlocal, or simply genuine nonlocal, and contains correlations which are stronger than any bipartite nonlocal correlations, even those which are signaling (since they would allow instant communication between separated parties \(^7\)).

The extension of Svetlichny inequalities to the case of \( N \) parties was accomplished in [SS02, CGP+02]. In general, a distribution probability is genuine nonlocal if and only if it cannot be written as the convex combination of any hybrid terms

\[
P_{A:B}(ab|xy) = \int d\lambda \omega(\lambda) P(a|x, \lambda) P(b|y, \lambda).
\]

(2.61)

where \( A \) : \( B \) is a splitting of the system into two subsystems, and \( a = a_1 \ldots a_k \) and \( b = a_{k+1} \ldots a_N \) represent the outcomes of local measurements \( x = x_1 \ldots x_k \) and \( y = x_{k+1} \ldots x_N \). Indeed, it is not hard to see that if there exist nonlocal correlations across any bipartition of the system, all the \( N \) parties must be jointly nonlocally correlated. Examples of \( N \)-partite entangled states that contain genuine nonlocality, in the sense that local measurements on the parties yield genuine nonlocal correlations, are the generalized GHZ-states

\[
|GHZ\rangle = \frac{1}{\sqrt{2}}(|00\ldots0\rangle + |11\ldots1\rangle),
\]

(2.62)

\(^7\) See section 2.3.5.
which maximally violate the Mermin-Svetlichny inequalities [SS02, CGP+02], and the generalized W-states

$$|W_N\rangle = \frac{1}{\sqrt{N}}(|0\ldots0\rangle + \ldots + |10\ldots0\rangle),$$  

(2.63)

which are only able to violate them by a very small amount [BBGP09].

Between the extreme cases of complete locality and genuine nonlocality, one can classify a probability distribution as $k$-nonlocal when it can be described by hybrid models whose nonlocal correlations involve a maximum of $k \leq N$ parties. Of course, the full characterization of multiparticle nonlocality requires studying every possible splitting of the parties, which soon becomes unfeasible with $N$. However, considering particular cases already provides some insight: in [JLM05, BBGP09], the authors considered hybrid models with specific groupings and communication patterns between the parties in order to characterize multiparticle nonlocality.

**Relation between Multipartite Entanglement and Multipartite Nonlocality**

How are entanglement and nonlocality related in the multiparticle scenario? The first link is that nonlocality in some partition of the system is only possible in the presence of entanglement across that same partition. Take a state $\rho$, separable in the partition defined by the grouping of parties $S = s_1, \ldots, s_m$, where $m \leq N$.

$$\rho = \sum_{i=1}^{k} p_i \tilde{\rho}_i^1 \otimes \tilde{\rho}_i^2 \otimes \ldots \otimes \tilde{\rho}_i^m$$  

(2.64)

then, by the same reasoning as in the bipartite case, it directly follows that it has a hybrid model which considers the same partition $S$. As expected then, nonlocality implies entanglement with full generality.

In the opposite direction, the bipartite scenario shows that pure entanglement guarantees nonlocality (by Gisin’s theorem), but the same does not hold in the mixed-state case. The results of [PR92] extend Gisin’s theorem [Gis91] into the multiparticle scenario. For that, the authors started by introducing generalized Bell inequalities for $N$-partite quantum states, which test the locality of the bipartite states

$$\rho_2 = \frac{M_{N-2}\rho M_{N-2}^\dagger}{\text{tr}_{N-2}(M_{N-2}\rho M_{N-2}^\dagger)}$$  

(2.65)

obtained after local measurements, $M_{N-2}$, on $N - 2$ parties of $\rho$. Then, it is shown that for any pure multiparticle entangled state $\rho = |\psi\rangle\langle\psi|$, there exist a local projections of the $N - 2$ parties that leave the remaining two systems in a pure entangled state. Consequently, for every $|\psi\rangle$ there is always one outcome of a projective measurement $M_{N-2}$, for which $\rho_2 = |\psi_2\rangle\langle\psi_2|$ is pure and entangled, hence it is nonlocal by Gisin’s theorem.

Notice that this procedure is very similar to the one used for nonlocality with sequential measurements (see section 2.3.2). In that case, the violation of a Bell inequality with sequential measurements did not imply nonlocality in
the standard Bell framework. For that, it would be necessary that the measurements on the subgroup that prepares the state (which will violate the Bell inequality), did not depend on the observables used in Bell test. In the case of sequential measurements, time-ordering of the events guarantees such independence. However, now, the fact that local measurements are performed on spacelike-separated systems is sufficient: the independence comes directly from the fact we assume that parties cannot signal arbitrarily fast. This means that Popescu and Rohrlich’s result [PR92] directly implies that any pure entangled multipartite state violates a standard Bell inequality. This fact was not mentioned in the original paper, which probably led to some misunderstanding on the community relative to the generality of such results (for instance, in 2004, the goal of [CWKO04] is to generalize Gisin’s theorem for three-qubit systems).

Note that given Gisin’s construction, it is even trivial to construct the multipartite Bell inequality which will be violated by any multipartite pure entangled state. Here, for simplicity, I consider a genuinely pure entangled tripartite state $\psi_{ABC}$ (the extension for the general case is straightforward). From [PR92], we know that there exist projective measurements on party $C$, which project $\psi_{ABC}$ on a pure entangled state $\psi_{AB}$ with some probability $P(c'|z')$. Then, $\psi_{ABC}$ violates a generalized inequality, conditioned on the outcome $c'$,

$$\sum_{abxy} c_{abxy} P(ab|xy, c'z') \leq k.$$  

(2.66)

We already know that $c_{abxy}$ are the coefficients of the CHSH-inequality [Gis91]. Now, in general, a probability distribution has the property

$$P(abc|xyz) = P(ab|xy, cz)P(c|xyz),$$  \hspace{1cm} (2.67)

but, since the physical probability distributions of outcomes are considered to be no-signaling, we have

$$P(c|xyz) = P(c|z).$$  \hspace{1cm} (2.68)

Then, we can simply use this identity to construct a tripartite Bell inequality from (2.66),

$$\sum_{abxy} c_{abxy} P(ab|xy, c'z')P(c'|z') = \sum_{abxy} c_{abxy} P(abc'|xyz') \leq kP(c'|z').$$  \hspace{1cm} (2.69)

This is of course a particular case of the standard form

$$\sum_{abcxyz} c_{abc|xyz} P(abc|xyz) \leq k$$  \hspace{1cm} (2.70)

where Charlie only needs to measure the observable $z = z'$ and all the coefficients of the Bell inequality associated to outcomes different from $c'$ are zero. An important remark on this multipartite extension of Gisin’s theorem is that it obviously provide no information on the kind of nonlocality contained in the multipartite pure entangled states.

And to finish, it is also known that genuine entanglement does not imply nonlocality under projective measurements in the multipartite scenario. Indeed, in [TA06], Acín and Toth constructed a local model for projective measurements on three-qubit states invariant under $U \otimes U \otimes U$ unitary transformations. This
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defines a two-parameter family of states, which includes a region of genuine tripartite and distillable states which are, nevertheless, completely local (according to (2.58)) [TA06].

In the multipartite scenario, there exist no studies on the possibility of activation of nonlocality, analogous to those in the bipartite case. Recall that there the standard definition of Bell tests was extended to include sequential measurements, collective measurements or interaction with ancillas. However, it is clearly an open question whether every multipartite entangled state contains some sort of nonlocality. The scenario is now even more interesting due to the fact that one could address the study to different kinds of nonlocal correlations.

GHZ paradox and multipartite EPR-2 approach

The GHZ-paradox is a quantum paradox originally introduced for three-particle systems, which arises from the nonlocal nature of quantum correlations [GHSZ90]. It is interesting since it shows a contradiction between local and quantum theories without requiring the use of Bell inequalities. It shows that according to the quantum model, there are outcomes of local measurements on quantum states, occurring with probability 1, which are incompatible with a completely local description. Performing the link with the idea underlying the (bipartite) EPR-2 decomposition (2.52), it means that no fraction of the outcome statistics can be reproduced by a completely local model (2.58). Therefore, any quantum states exhibiting a GHZ-like paradox has completely local weight \( p_L \) equal to zero.

I proceed now by introducing the original GHZ-paradox for three particle states, introduced in [GHSZ90], but following the simplified version of Mermin [Mer90b]. Consider then a system of three spin-1/2 particles prepared in the GHZ state

\[
|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle),
\]

(2.71)

shared by separated observers Alice, Bob and Charlie. This state has the property of being a common eigenstate to the following triplets of local observables

\[
\sigma^A_x \otimes \sigma^B_y \otimes \sigma^C_y
\]

(2.72)

\[
\sigma^A_y \otimes \sigma^B_x \otimes \sigma^C_y
\]

(2.73)

\[
\sigma^A_y \otimes \sigma^B_y \otimes \sigma^C_x
\]

(2.74)

and, since these are mutually commuting, (2.71) is also an eigenstate of

\[
\sigma^A_x \otimes \sigma^B_x \otimes \sigma^C_x
\]

(2.75)

where the \( \sigma_i, i = x, y, z \), represent the Pauli operators. For the first set of observables (2.72)-(2.74), the eigenstate (2.71) corresponds to the eigenvalue 1, while (2.75) is associated to the eigenvalue -1. One can now build a Bell experiment where each party measures the observables \( \sigma_x \) and \( \sigma_y \) on their local spin-system, obtaining one of the possible outcomes \( \pm 1 \). According to quantum theory, the eigenvalues of the operators define the outcomes of the experiment, which occur with certainty (due to the perfect correlation properties of the state). In order to check whether these outcomes are compatible with a local theory, one defines \( s_i^\theta = \pm 1 \) as the value assigned by the hidden-variable to the
2.3. QUANTUM NONLOCALITY

observable $s_i^O$, and these values must satisfy all the following conditions

\[
\begin{align*}
    s_x s_y s_y &= 1 & (2.76) \\
    s_x s_y s_y &= 1 & (2.77) \\
    s_y s_y s_x &= 1 & (2.78) \\
    s_x s_x s_x &= -1. & (2.79)
\end{align*}
\]

But this is easily seen to be impossible: multiplying the first three products
of outcomes, (2.76)-(2.78), one obtains $s_x s_x s_x = 1$ which is always in contradic-
tion with (2.79). Then, there is no possible assignment of values $\pm 1$ to the
classical observables $s_i^0$ which is in agreement with results of this given experi-
ment.

GHZ-like paradoxes can be obtained for any number of parties, proving that
in the general multipartite case there are examples of quantum distributions
incompatible with a local model in the strongest sense: no fraction of quantum
statistics can be reproduced by local statistics. In particular, this is true for
correlations arising from measurements on multipartite GHZ-states (2.62).

As a final remark, notice that one can also prove that there exist multipartite
quantum states with $p_L = 0$ using the derivation of [BKP06]. For that, it is
required that the quantum state violates a standard multipartite Bell inequality
(not necessarily Svetlichny), reaching the maximum nonlocal violation of the
inequality.

2.3.5 Local, Quantum and No-Signaling Theories

The EPR-2 approach to quantum nonlocality considers the possibility that the
observed quantum correlations could arise as an effective phenomenon: physical
particles would belong to one of two factions, one composed of particles having
classical (local) behavior and another where particles would be nonlocal and
not necessarily ruled by quantum theory. What are then these generic nonlocal
thories? How can we distinguish them from quantum theory?

In 1994, Popescu and Rohrlich [PR94] raised the question whether quantum
 correlations could arise from two essential physical principles: nonlocality
and relativistic causality. According to the second, instantaneous commu-
nication between distant locations is forbidden, which is known as the no-signaling
principle. In terms of the probability distributions $P(ab|xy)$, this imposes the
conditions

\[
P_{NS}(b|xy) = \sum_a P_{NS}(ab|xy) = P_{NS}(b|y), \forall b,x,y
\]

and the same for the other party (see also Appendix A). Surprisingly, they found
this imposes no restrictions on the possible violation of the CHSH inequality:
there exist no-signaling non-local correlations able to achieve the value 4, which
corresponds to the algebraic limit of the expression (2.25). These intriguing
nonlocal correlations are

\[
P_{PR}(ab|xy) = \begin{cases} 
1/2 & \text{if } a \oplus b = xy \\
0 & \text{otherwise} \end{cases}
\]

and the nonlocal device that produces them became known as PR-box. Note
that the no-signaling condition (2.80) is guaranteed by the fact that locally the
outcomes are completely random, therefore independent of any measurement setting. Since relativistic causality is not sufficient to single out quantum correlations among general nonlocal correlations, it is a fundamental open question to find such principle(s). In Appendix A it is shown that local, quantum and no-signaling correlations define convex sets in a multidimensional probability space. This geometrical description turns out to be very useful: finding the set of quantum correlations corresponds, in this picture, to define principles that exclude all points of the set of no-signaling correlations not belonging in the quantum set. A possible approach is to identify phenomena which can only be observed in the presence of supra-quantum correlations. The non-observation of these phenomena can always be imposed as a physical principle that rules out the corresponding supra-quantum theories.

Interestingly, the first result along this line associates the availability of PR-boxes (2.81) with strange consequences at level of information and computation theory concepts, rather than on physical phenomena. In [vD05], van Dam considered the distributed computation of a given Boolean function \( f(x, y) \), where the input bit-strings \( x \) and \( y \) are, respectively, held by separated parties Alice and Bob. The communication complexity associated to the function \( f \) is defined as the minimum amount of information that the parties need to exchange in order to compute its value. The inner product function,

\[
f(x, y) = x \cdot y = x_1 \cdot y_1 + \ldots + x_N \cdot y_N
\]

is particularly interesting because it has maximum communication complexity: for input strings of length \( N \), Alice needs to send a minimum of \( N \) bits to Bob (that is, her entire bit-string) in order to evaluate the function, even if the parties share quantum resources [CvDNA98]. However, if Alice and Bob share \( N \) copies of PR-boxes (2.81), and choose as the input of each nonlocal box \( i \) to be the value of their input bit \( i \), they obtain

\[
f(x, y) = \bigoplus_{i=1}^{N} x_i \cdot y_i = \bigoplus_{i=1}^{N} a_i \oplus b_i = (\bigoplus_{i=1}^{N} a_i) \oplus (\bigoplus_{i=1}^{N} b_i) \equiv \alpha \oplus \beta,
\]

where \( \alpha \) and \( \beta \) correspond to local bits arising solely from the use of their shared no-signaling resources. But now, to evaluate \( f \), it suffices for Alice to send her bit \( \alpha \) to Bob, which means that the communication complexity becomes trivial [vD05]. Moreover, any Boolean function of \( N \) bits can be mapped into the Inner Product function of input size \( 2^N \) [vD05], which implies that the availability of PR-boxes would collapse the communication complexity of any Boolean function. This is seen as an extremely unlikely scenario by the computer science community, and therefore a good reason to exclude these extremal no-signaling correlations [vD05].

In [BBL+06], van Dam’s result is generalized for a larger set of no-signaling bipartite correlations. For that, Brassard and co-authors considered a probabilistic version of communication complexity. The distributed computation of some Boolean function is trivial if, with less than some critical amount of communication \( c_{\text{min}} \), it is possible to perform it with a success probability higher than \( 1/2 \). For the probabilistic computation of the Inner Product function (2.82), both local and classical resources require the same amount of communication \( c \geq c_{\text{min}} \). However, noisy PR-boxes, which are the admixture of PR-boxes
2.3. QUANTUM NONLOCALITY

with uncorrelated local noise,

\[ P_{\epsilon}^{PR} = \epsilon P^{PR} + (1 - \epsilon) P^{I}, \]  \hspace{1cm} (2.84)

allow trivial communication for \( \epsilon \geq 0.91 \). Note that (2.84) defines a family of distributions represented in Fig. 2.2, where \( \epsilon_Q = 0.85 \) corresponds to the entrance in the quantum region through the point of maximal violation of the CHSH inequality. Therefore, these results give support for the non-availability of noisy PR-boxes in the range of parameters \( 0.91 \leq \epsilon \leq 1 \). It should be noted that although obviously very interesting, such results do not exclude more than a portion of zero-measure of the set of bipartite supra-quantum correlations (see figure 2.2).

In [BS09], the combination of two local protocols shows that trivial communication complexity would also be observed in a more significant (of non-zero measure) portion of the set of supra-quantum correlations. On one hand, the authors construct a distillation protocol for nonlocality, inspired by [FWW09], which increases the fraction of PR-boxes (2.81) when taking two copies of the mixture \(^8\)

\[ P_{\delta}^{PR} = \delta P^{PR} + (1 - \delta) P^c \] \hspace{1cm} (2.85)

where \( P^c \) are completely correlated local distributions of the kind

\[ P^c(ab|xy) = \begin{cases} 
1/2 & \text{if } a \oplus b = 0 \\
0 & \text{otherwise}.
\end{cases} \] \hspace{1cm} (2.86)

Observe that these correspond to local deterministic strategies (see figure 2.2), and therefore the family of states (2.86) is defined along the lines of the no-signaling polytope that connect the vertices of the local polytope with the PR-boxes. Since this distillation protocol does not require communication, any supra-quantum correlation (2.85) with \( \delta > 0 \) also yields trivial communication complexity. They consider then the depolarizing protocol from [MAG06], which maps any nonlocal correlation to isotropic correlations, i.e. of the kind (2.84), with the same amount of CHSH violation. Geometrically, it means that the boxes along a parallel line are mapped into the same corresponding point, along the isotropic boxes vertical line (see figure 2.2). With this, it is possible to associate the availability of the majority of bipartite supra-quantum correlations with the unlikely collapse of communication complexity [BS09].

Shortly after, Pawłowski and co-authors [PloPK+09] define a new physical principle, Information Causality, which holds for any bipartite classical and quantum correlation, but violated by most bipartite supra-quantum correlations. Information causality can be seen as a generalization of the no-signaling principle: it states that if Alice sends \( m \) bits to Bob, the shared information between them cannot be larger than \( m \), regardless of the physical resources they might share. It is known that all nonlocal non-quantum correlations above the Tsirelson’s bound [PloPK+09], as well as some below [ABPS09], violate this principle. It is not known whether it could be extended to all supra-quantum correlations and finally recover the quantum boundary.

In a more general framework, where Alice and Bob each have access to more than two observables, the task known as nonlocal computation [LPSS07] also

\(^8\)In fact, the protocol also distills a broader class of nonlocal boxes, namely \( P^{PR}_{\delta,\gamma} = \delta P^{PR} + \gamma P^{T} + (1 - \delta - \gamma) P^c \), where \( P^{T} \) is a PR-box with \( a \oplus b = x \cdot y + 1 \).
marks a sharp difference between classical/quantum resources and general no-signaling ones (see also App. C.4). It can be seen as a distributed computation task between two parties, where the possibility of exchanging information is replaced by correlation between the input bits sent to Alice and Bob. It goes as follows. Each party receives a $n$-bit string, $x$ for Alice and $y$ for Bob, such that the logical sum is the input $z = x \oplus y$ of a given Boolean function $f$. The goal is to evaluate $f(z)$ by outputting bits $\alpha$ (Alice) and $\beta$ (Bob) whose sum provides the value of the function, $f(x \oplus y) = \alpha \oplus \beta$. The inputs $z$ are distributed according to some distribution $p(z)$ known to the parties, but they gain no information about their partner’s input upon receiving their bit-string since, for every $z$, $x$ and $y$ are uniformly distributed: $p(x, y | z) = 1/2^n$, for $x \oplus y = z$. It happens that for this task, if Alice and Bob share classical or quantum correlations, they have the same success probability of guessing the value of the function. In fact, the best they can do is to perform the linear approximation to the Boolean function $f$, that is, to output $f_1(z) = c \cdot z \oplus \delta$. It follows that if the function is linear, any physical resource provides perfect nonlocal computation; but for higher order functions, quantum and local resources can only achieve bounded success probability. On the contrary, if parties share the generalized PR-boxes

$$P_{PR}^{ab | xy}(ab) = \begin{cases} 1/2 & \text{if } a \oplus b = f(x \oplus y) \\ 0 & \text{otherwise} \end{cases}$$

they perform perfect nonlocal computation [LPSW07]. Mind that this result differs from the previous ones, in the sense that it defines a task which is associated to a Bell inequality (see Appendix C.4 for more details). The fact that local and quantum correlations perform equally well means that the Bell inequality is tangent to both the local polytope and the quantum set; the existence of supra-quantum advantage implies that there exist a no-signaling violation of the Bell

Figure 2.2: Representation of a region of the no-signaling space for two parties with two dichotomic observables. It contains a facet of the local polytope $L$, a portion of the convex set $Q$ of quantum correlations and a portion of the polytope of no-signaling distributions $NS$. 
inequality. However, it is clear that nonlocal computation, which corresponds to a linear Bell inequality tangent to the local set, can never single out the entire quantum border, which is defined by a nonlinear equation.

Although I mainly focused on how concepts from theory of information and computation can be used to justify quantum nonlocal correlations, numerous results show that supra-quantum correlations would also have a strong impact at the level of physical phenomena. Just to name a few, it is known, for instance, that some no-signaling theories would not allow entanglement swapping [SPG06, SBP09, SB10], teleportation or dense coding [Bar07, SB10]. Also, all reversible dynamics in maximally nonlocal theories are trivial [GMCD10]. This arises from a very interesting observation: there is a trade-off between the allowed states of a theory and its allowed dynamics [Bar07]. To finish, I would like to mention the recently proposed macroscopic locality, which states that any physical theory should recover classical correlations in the continuum limit, i.e., in the presence of large number of particles and measurement devices unable to resolve individual particles [NW10]. This single principle together with no-signaling is enough to recover the border of the quantum set of full bipartite correlations, with two dichotomic observables per party. Unfortunately, this does not hold when considering partial correlations [NW10].

We have seen that although there is an intense interest and effort to find physical principles that would identify quantum correlations, this is still an open question even in the simplest nontrivial case of bipartite correlations. Moreover, up to my knowledge, no results are known at the level of multipartite nonlocal correlations.
Chapter 3

Structural approximations to positive maps and entanglement breaking channels

Structural approximations to positive non-completely-positive maps are approximate physical realizations of these non-physical maps. They find applications in the design of direct entanglement detection methods. In this chapter, we prove that many of these approximations, in the relevant case of optimal positive maps, define an entanglement-breaking channel and, consequently, can be implemented via a measurement and state-preparation protocol. We also show that these results can be useful for the design of better and simpler direct entanglement detection methods.

3.1 Introduction

Entanglement is one of the most important, and presumably necessary, ingredients of quantum information processing [HHHH09]. For this reason there is a considerable interest both in theory and experiments in designing feasible and efficient ways of entanglement detection (see for instance [GT09] for a review). Although there exist efficient methods to identify several classes of entangled states, detecting any entangled state requires either (i) performing quantum tomography to the quantum state and then applying the criterion based on positive maps or (ii) directly measuring the expected value of a suitable entanglement witness (see section 2.1).

As we have seen in section 2.1.2, although a single positive map detects a much larger set of quantum states than its correspondent entanglement witness, the fact that they are not physical operations and therefore require prior state estimation is a drawback in its implementation. This is specially relevant in the case of large systems, where the number of real parameters to estimate in a $d_A \otimes d_B$ quantum state is $d_A^2 d_B^2 - 1$, many of them unnecessary for entanglement detection. The goal of [HE02] is exactly to find a structural physical
approximation (see Section 2.2.2) to these positive maps in order to bypass the state estimation process. The idea is to mix a positive map $\Lambda$ with some simple completely positive (CP) map, making the mixture $\tilde{\Lambda}$ completely positive. The resulting map can then be realized in the laboratory and its action characterizes the entanglement of the states detected by $\Lambda$. In the particular example studied in [HE02] this idea has been applied to the partial transposition map $\Lambda = \mathbb{1} \otimes T$. Although experimentally viable, this method is not easy to implement since, at least in its original version, it requires highly nonlocal measurements: subsequent applications of $\tilde{\Lambda}$ followed by optimal spectrum estimation. A more detailed discussion on this entanglement detection scheme is given in Section 3.4.

In this chapter we address the question of implementation of structural physical approximations to positive maps through generalized measurements. In particular, we study structural approximations to maps $\Lambda: \mathcal{B}(\mathcal{H}_A) \rightarrow \mathcal{B}(\mathcal{H}_B)$, obtained through minimal admixing of white noise:

$$\tilde{\Lambda}(\rho) = p \frac{1}{d_B} + (1 - p) \Lambda(\rho).$$

(3.1)

By minimal we mean the smallest noise probability $0 < p < 1$ for which $\tilde{\Lambda}$ becomes completely positive. Using the Jamiołkowski isomorphism (2.8), we can express this in terms of entanglement witnesses. The structural physical approximation consists in mixing a witness $\tilde{W}$ with the amount of white noise necessary for the resulting $\tilde{W}$ to be a quantum state, that is,

$$\tilde{W}_\Lambda = \mathbb{1} \otimes \tilde{\Lambda}(\Phi_{d_B}^+) = \frac{p}{d_Ad_B} \mathbb{1} + (1 - p)W_\Lambda$$

(3.2)

such that

$$\tilde{W}_\Lambda \geq 0.$$  (3.3)

This condition is equivalent to $\tilde{\Lambda}$ being completely positive [HSR03].

Now the key question is when such channel $\tilde{\Lambda}$ can be implemented through generalized measurements. We have seen in Sec. 2.2.1 that this is possible if and only if the structural approximation $\tilde{\Lambda}$ corresponds to an entanglement-breaking (EB) channel. Moreover, from the decomposable representation of $\tilde{W}$, we directly obtain the explicit measurement and preparation protocol (according to equations (2.17) and (2.18)).

The main subject of this chapter is then the following conjecture:

**Conjecture:** Structural physical approximations to optimal positive maps correspond to entanglement-breaking channels. Equivalently, structural physical approximations to optimal entanglement witnesses $W$ are given by separable states.

---

1. A different approach to avoid state tomography was taken in [ASan08]. Here the authors use the fact that every positive map $\Lambda$ can be written as the difference of two completely-positive maps, $\Lambda = \Lambda_1 - \Lambda_2$, to obtain a direct entanglement-detection method derived from the $\mathbb{1} \otimes \Lambda(\rho) \geq 0$ criterion.

2. In fact, it was proved in [Fiu02] that the example analyzed in [HE02] for the case of two-qubit states has such realization. But notice that this example does not entirely fit into our framework since the initial map is not even positive. See discussion in Sec. 3.4.
We prove the above conjecture for decomposable maps, which are simple to characterize but unfortunately unable to detect PPT states. These are only detected by non-decomposable maps, for which no general structural characterization is known. Consequently, for this case, we are limited to consider specific examples. We study some of the most famous non-decomposable (including those where optimality is not proven) and show that the conjecture holds.

The importance of our results is twofold: i) if the conjecture is true, structural physical approximations to optimal maps admit a particularly simple experimental realization – they correspond to generalized measurements; ii) the results shed light on the geometry of the set of entangled and separable states.

This chapter is organized as follows. In Sec. 3.2 we concentrate on decomposable maps. There we prove that the conjecture is true for the transposition map, the only optimal map in this class. We also show that the assumption of optimality is essential: we provide an example of non-optimal $2 \otimes 2$ witness whose structural approximation leads to an entangled state. Section 3.3 is devoted to non-decomposable positive maps. We start the discussion by analyzing Choi’s map, one of the first examples of non-decomposable maps. Then, we study a positive map based on unextendible product basis [BDM99]. Finally, we end this section with an analysis of the Breuer-Hall map [Bre06, Hal06], which can be understood as the non-decomposable version of the reduction criterion [HH99]. Here symmetry methods turn out to be indispensable, and we are led to introduce and study a new family of states – unitary symplectic invariant states. The most technical details of this study are given in Appendix B, where, as a byproduct, we show that this family also includes a region of bound entanglement. Finally, we study the physical approximation to partial transposition, as this map is used in the direct entanglement detection method proposed in [HE02]. In the latter case the analysis is again made possible due to symmetry arguments, in particular the unitary $UUVV$ symmetry [VW01, LMD08]. We end with the conclusions in section 3.5.

### 3.2 SPA to Decomposable Maps

This section is devoted to the study of the conjecture for decomposable maps. We prove the conjecture for the transposition map, which is the single optimal decomposable map. Then, we provide an example of a witness on a $2 \otimes 2$ Hilbert space that shows that the assumption of optimality is essential in the conjecture.

#### 3.2.1 Transposition Map

We consider the transposition map $T : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$, in arbitrary dimension $\mathcal{H} \cong \mathbb{C}^d$. Applying equations (2.10) and (3.2), we directly obtain the SPA to the associated witness $W_T$, which turns out to a Werner state on $\mathcal{H} \otimes \mathcal{H}$

$$
\bar{W}_T = p \frac{1}{d} + (1 - p) \frac{F}{d}.
$$

Here $F$ is the flip operator, such that $F|\psi\rangle|\phi\rangle = |\phi\rangle|\psi\rangle$, and the noise parameter is

$$
p = \frac{d}{d + 1}.
$$
CHAPTER 3. STRUCTURAL APPROXIMATIONS TO POSITIVE MAPS AND ENTANGLEMENT BREAKING CHANNELS

To check the separability of $\tilde{W}_T$, we use the fact that the PPT criterion is necessary and sufficient for Werner states. For every $0 \leq p \leq 1$, we have

$$\tilde{W}_T^\Gamma = \frac{p}{d^2} \mathbb{1} + (1 - p) P_+ \geq 0,$$

hence $\tilde{W}_T$ becomes separable at the point it becomes a state\(^3\). This implies that the structural approximation to transposition is entanglement-breaking and the conjecture holds.

Now we present a specific preparation and measurement protocol that replaces the action of the transposition map. For that, we must first find a decomposable representation of (3.4), which is possible using its invariance under unitary $UU$ transformations [Wer89]. According to it, Werner states have the integral representation [Wer89]

$$\rho_W = \int dU (U \otimes U) \rho (U^\dagger \otimes U^\dagger),$$

where $dU$ corresponds to the Haar measure over the unitary group (for more details, see Subsection 2.3.2). Since Werner states are spanned by the operators \(\{1, F\}\), they are completely defined by the parameter \(\langle F \rangle = \text{tr} (\rho_W F)\) [Wer89]. In our case, \(\text{tr}(\tilde{W}_T F) = \text{tr}(\rho F) = 1\) and therefore the state can be written in the form (3.7) with \(p = |00\rangle \langle 00|\). With this, we find an explicit expression for $T$ in the Holevo form (2.16).

$$\tilde{W}_T = \int dU |v_U\rangle \langle v_U| \otimes |w_U\rangle \langle w_U|$$

with $|v_U\rangle = |w_U\rangle = U|0\rangle$. Notice that (3.8) is a continuous version of (2.17), where the discrete set of states \(\{|v_k\rangle, |w_k\rangle\}\) is replaced by a continuous set \(\{|v_U\rangle, |w_U\rangle\}\) and the probability $p_k$ is replaced by the probability distribution $dU$. According to Eq. (2.18), $\tilde{T}$ can be written as

$$\tilde{T}(\rho) = \int dU |\bar{w}_U\rangle \langle \bar{w}_U| \text{tr} \left[ (d|v_U\rangle \langle v_U|) \rho \right],$$

where we used the invariance of the integral under conjugation. This approximation has a clear intuitive explanation. Given an unknown state, first one tries to estimate it in an optimal way using the covariance measurement defined by the infinite set of operators \(\{M_U = d|v_U\rangle \langle v_U|\}\), distributed according to the Haar measure. If the measurement outcome corresponding to $|v_U\rangle$ is obtained, the state $|v_U\rangle |v_U\rangle^T = |w_U\rangle \langle w_U|$ is prepared. Finally, it is important to mention that the map defining the depolarization process $\mathcal{D}_{UU}$ can also be implemented by the finite set of unitary operators \(\{p_k, U_k\}\) of [DCLB00], which in our case leads to a measurement with a finite number of outcomes.

3.2.2 SPA to positive maps are not always entanglement-breaking

Now we provide an example of non-optimal positive map which does not lead to an entanglement-breaking channel. This confirms the need to include the optimality requirement in the conjecture. Our example considers an (unnormalized)\(^3\)Werner states are actually defined for a parameter $d/(d+1) \leq p \leq d/(d-1)$, being entangled for $p > d^2/(d^2-1) > 0$
3.3. NON-DECOMPOSABLE MAPS

entanglement witness on a \(2 \otimes 2\)-dimensional Hilbert space. Take \(P_1\) and \(P_2\) to be positive rank-one operators of the form

\[
P_1 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & b & b & 0 \\
0 & b & b & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad P_2 = \begin{bmatrix}
a & 0 & 0 & a \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
a & 0 & 0 & a \\
\end{bmatrix}
\]

(3.10)

with real positive \(a\) and \(b\), which define the non-optimal witness (see Sec. 2.1.3)

\[
W = P_1 + P_2^\Gamma = \begin{bmatrix}
a & 0 & 0 & 0 \\
0 & b & b + a & 0 \\
0 & b + a & b & 0 \\
0 & 0 & 0 & a \\
\end{bmatrix}
\]

(3.11)

According to (3.2), the structural physical approximation to \(W\) is

\[
\tilde{W} = \begin{bmatrix}
a' + c & 0 & 0 & 0 \\
0 & b' + c & b' + a' & 0 \\
0 & b' + a' & b' + c & 0 \\
0 & 0 & 0 & a' + c \\
\end{bmatrix}
\]

where \(a' = (1 - p)a\), \(b' = (1 - p)b\), \(c = p/4\), and the minimum amount of noise added is

\[
p = \frac{4a}{4a + 1}.
\]

(3.12)

However, according to the PPT criterion, the state (3.12) is entangled for

\[
p_{\text{ent}} < \frac{4b}{4b + 1}.
\]

(3.13)

Then, if the noise parameter is in the range where \(b > a\), the structural approximation is not entanglement-breaking and this simple example shows that the assumption of optimality is essential in the conjecture.

3.3 Non-decomposable maps

In this section, we move to non-decomposable maps. We first consider the Choi map, which is one of the first examples of a non-decomposable positive map. After this, we study positive maps derived from unextendible product bases. Finally, we analyze a recently introduced positive map, the Breuer-Hall map.

3.3.1 Choi map

Consider the non-decomposable map \(\Lambda_C : B(C^3) \rightarrow B(C^3)\) introduced by Choi [Cho75]

\[
\Lambda_C(\rho) = \frac{1}{2} \left( -\rho + \sum_{i=0}^{2} \rho_i (2|i\rangle\langle i| + |i - 1\rangle\langle i - 1|) \right),
\]

(3.14)

where \(|i\rangle\) is a fixed basis of \(C^3\) and the summation is modulo 3. According to Eq. (3.2), the state \(\tilde{W}_C\) associated with the structural approximation \(\tilde{\Lambda}_C\) reads

\[
\tilde{W}_C = p \frac{\mathbb{1}}{9} + \frac{1 - p}{6} \left( \sum_{i=0}^{2} (2|ii\rangle\langle ii| + |i, i - 1\rangle\langle i, i - 1|) - 3\Phi_i^+ \right).
\]

(3.15)
with \( p = 3/5 \). The state \( \tilde{W}_C \) is separable since it can be written as the convex combination of (unnormalized) product states
\[
\tilde{W}_C = \frac{1}{15} (\sigma_{01} + \sigma_{12} + \sigma_{02} + \sigma_d) ,
\]
(3.16)
with \( \sigma_d = |02\rangle\langle02| + |10\rangle\langle10| + |21\rangle\langle21| \) obviously separable and states \( \sigma_{ij} \)
\[
\sigma_{ij} = \mathbb{1} - |ii\rangle\langle jj| - |jj\rangle\langle ii| .
\]
(3.17)
These are actually defined on \( 2 \otimes 2 \) subspaces, spanned by \( \{ |ii\rangle, |ij\rangle, |ji\rangle, |jj\rangle \} \), so it is enough to use the PPT condition to check their separability.

Although Choi’s map is not proven to be optimal, its structural approximation is entanglement-breaking and therefore this example can never disprove our conjecture.

### 3.3.2 UPB Map

Here we consider the non-decomposable witnesses that detect the PPT entangled states defined through unextendible product basis (UPB) [BDM+99]. First, mind that an UPB in an arbitrary space \( \mathcal{H}_A \otimes \mathcal{H}_B \cong \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} \) consists of a set of \( n < d_A d_B \) orthogonal product states, \( \{ |v_i\rangle = |x_i\rangle|y_i\rangle \}_{i=1}^n \), such that there is no product state orthogonal to them. It is then impossible to extend this set into a full product basis. The state defined as the uniform mixture on the space orthogonal to the UPB
\[
\rho_{\{v\}} = \frac{1}{d_A d_B - n} \left( \mathbb{1}_{AB} - \sum_{i=1}^n |v_i\rangle\langle v_i| \right) .
\]
(3.18)
is entangled by definition, and PPT since its partial transposition defines a new quantum state.

A witness that detects such states is
\[
W_{\{v\}} = \frac{1}{n - \epsilon d_A d_B} \left( \sum_{i=1}^n |v_i\rangle\langle v_i| - \epsilon \mathbb{1}_{AB} \right) ,
\]
(3.19)
where \( \epsilon > 0 \) can be taken arbitrarily small [BDM+99]. The structural approximation to the witness reads
\[
\tilde{W}_{\{v\}} = \frac{1}{n - \epsilon d_A d_B} \left( np - \epsilon d_A d_B \right) \mathbb{1} + (1 - p) \sum_{i=1}^n |v_i\rangle\langle v_i| .
\]
(3.20)
where the value of the noise parameter \( p \), such that \( \tilde{W}_{\{v\}} \) is positive, depends on the value of \( \epsilon \). But we know that in the point where \( \tilde{W}_{\{v\}} \) defines a quantum state, it must be separable since it is the convex combination of separable states. Therefore, structural approximations to witnesses arising from UPBs are entanglement-breaking. It is not known whether these witnesses are optimal, but as in the case of Choi’s map, this example can never disprove our conjecture.

We can use the fact that \( \tilde{W}_{\{v\}} \) is already in a product state form to directly write the state resulting from the action of the channel \( \Lambda_{\{v\}}(\rho) \) in its Holevo representation (2.18). As mentioned, this gives the explicit construction of the measurement and state-preparation protocol approximating the action of the positive map.
3.3. Non-Decomposable Maps

3.3.3 Breuer-Hall Map

We are left with the case of the optimal non-decomposable Breuer-Hall map, recently introduced in [Bre06, Hal06], and which can be understood as the generalization of the reduction criterion [HH99] to the non-decomposable case. We are able to prove that this positive map satisfies the conjecture for any dimension where it is defined. For that, we analyze new families of quantum states, \( \rho_{SS} \) and \( \rho_{\bar{S}S} \), characterized by the invariance under \( S \otimes S \) and \( S \otimes \bar{S} \) transformations, respectively, where \( S \) is an unitary symplectic operator (see definition below). In the following, we concentrate on the structural approximation to the Breuer-Hall map, leaving the detailed study of these new families of states to Appendix B.

The reduction map is a decomposable non-optimal positive map, introduced in [HH99], which leads to a separability criterion that relates the structure of the bipartite state with the local reduced states. It reads

\[
\Lambda_{\text{red}}(\rho) = \text{tr}(\rho) \frac{1}{d} - \rho \tag{3.21}
\]

and any entangled state it detects is known to be distillable [HH99].

Recently, it was realized that for even dimension, \( d = 2n \geq 4 \), it is still possible to remove a positive component from the reduction map (3.21) without affecting the positivity of the resulting map [Bre06, Hal06]. The Breuer-Hall map is then defined according to

\[
\Lambda_{\text{BH}}(\rho) = \frac{1}{d-2} \left[ \text{tr}(\rho) \mathbb{1} - \rho - U \rho^T U^\dagger \right], \tag{3.22}
\]

where the term \( U \rho^T U^\dagger \) consists on the positive map given by the transformation of transposition by an operator \( U \), both skew-symmetric,

\[
U^T = -U \tag{3.23}
\]

and unitary. This provides a significant improvement since the Breuer-Hall map is non-decomposable and moreover is known to be optimal [Bre06].

Now, following our usual procedure, we obtain the structural approximation to the associated witness \( \tilde{W}_{BH} \).

\[
\tilde{W}_{BH} = \frac{1}{d-2} \left( \frac{d-2p}{d^2} \mathbb{1} - (1-p)\Phi_2^+ - \frac{1-p}{d} \left( \mathbb{1} \otimes U \right) \Phi \left( \mathbb{1} \otimes U^\dagger \right) \right), \tag{3.24}
\]

where \( p \) is the minimum amount of noise such that \( \tilde{W}_{BH} \) is positive. In order to find this parameter for arbitrary \( d \) we use the fact that (3.24) belongs to the family of \( SS \)-invariant states, where \( S \) is symplectic,

\[
S^T JS = J, \tag{3.25}
\]

for \( J \) skew-symmetric, and also unitary.

The proof of \( SS \)-invariance of (3.24) goes as follows. First of all, notice that in even dimension any skew-symmetric matrix \( J \) is non-degenerate (\( \text{det} J = 1 \)) and can be represented in the Darboux basis, where \( J \) takes the canonical form [Ham89]

\[
J = \bigoplus_{i=1}^n \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.26}
\]
In this basis, \( J \) is obviously unitary and therefore we will use it to represent our unitary skew-symmetric matrices: \( U = J \). To prove the \( SS \)-invariance of (3.24),

\[
S \otimes \bar{S} \tilde{W}_{BH} S^\dagger \otimes S^T = \tilde{W}_{BH},
\]

we just need to prove this result for every operator appearing in its decomposition. For \( \Phi_d^+ \) we use the fact that it is \( U \otimes \bar{U} \) invariant, for unitary \( U \) [HH99]. As for the operator \( \mathbb{1} \otimes JF \otimes J^\dagger \), it follows from the property

\[
\bar{S}J = JS
\]

(3.28) together with the \( U \otimes U \)-invariance of \( F \) [Wer89]. Consequently, any state in this family as the integral representation

\[
\rho_{SS} = \int dS (S \otimes \bar{S}) |\varphi\rangle \langle \bar{\varphi}| (S \otimes \bar{S})^\dagger
\]

(3.29) and lives in the Hermitian space spanned by the set of operators \( \{ \mathbb{1}, F^d, \Phi_d^+ \} \) (see derivation in Appendix B). It follows that normalized \( SS \) invariant states can be characterized by two parameters only: \( \langle F^d \rangle \) and \( \langle \Phi_d^+ \rangle \). For the operator \( \tilde{W}_{BH}(p) \), these expectation values are

\[
\langle \Phi_d^+ \rangle = \frac{p(d+1) - d}{d^2}
\]

(3.30)

\[
\langle F^d \rangle = \frac{p(d+1) - d}{d^2}
\]

(3.31)

Then, the family of operators \( \tilde{W}_{BH}(p) \) describes a line in the space of parameters \( \{ \langle F^d \rangle, \langle \Phi_d^+ \rangle \} \), which enters the region of quantum states through the point of coordinates \( \langle F^d \rangle, \langle \Phi_d^+ \rangle \) = (0, 0). (See Fig. B.1 in Appendix B minding that it is completely equivalent to the \( SS \)-invariance, after the relabeling of axis.) In the Appendix B we obtain the set of parameters for which \( \rho_{SS} \) is separable. In particular, we show that the point \((0,0)\) is an extremal point of this convex set. Therefore, we conclude that the structural approximation is entanglement-breaking and the conjecture holds.

Since the state \( \tilde{W}_{BH} \) has the critical noise parameter

\[
p = \frac{d}{d+1},
\]

(3.32)

it can be written according to the separable representation (3.29), with \( |\varphi\rangle = |\phi\rangle \otimes |\psi\rangle \) and

\[
|\phi\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle + |2\rangle + |3\rangle)
\]

(3.33)

\[
|\psi\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |2\rangle).
\]

(3.34)

The channel can then be written in its Holevo form,

\[
\tilde{\Lambda}_{BH}(\varrho) = \int dS |w_S\rangle \langle w_S| \text{tr} (d|v_S\rangle \langle v_S| \varrho),
\]

(3.35)

with \( |v_S\rangle = S|\phi\rangle \) and \( |w_S\rangle = S|\psi\rangle \). As we have seen previously, this provides an explicit measurement and state preparation protocol that replaces the action of \( \tilde{\Lambda}_{BH} \).

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3.4 Entanglement Detection via Structural Approximations

Before concluding, we would like to discuss the application of these ideas to the design of entanglement detection methods. Indeed, one of the main motivations for the introduction of structural approximations [HE02] was to obtain approximate physical realizations of positive maps, which can then be used for experimental entanglement detection. Notice that our results do not directly provide such a scheme. Although we are able to physically approximate optimal positive maps with measurement and state preparation protocols, we would still need to define a suitable criterion to identify the presence of entanglement. As we will now see, in the scheme proposed in [HE02], this criterion is directly derived from the separability criterion with positive maps.

The original proposal by [HE02] works as follows, see also Fig. 3.1. Given $N$ copies of an unknown bipartite state, $\varrho_{AB}$, the goal is to determine, without resorting to full tomography, whether the state is PPT. The idea is to apply the structural approximation to partial transposition to this initial state and estimate the spectrum (or more precisely, the minimal eigenvalue) of the resulting state using the optimal measurement for spectrum estimation described in [KW01]. Note that the structural approximation $\tilde{1}_{A} \otimes T$ “simply” adds white noise to the ideal operator $\varrho^T$. Thus, it is immediate to relate the spectrum of $(\tilde{1}_{A} \otimes T)(\varrho_{AB})$ to the positivity of the partial transposition of the initial state.

Inspired by the previous findings, we study in this section whether the structural approximation to partial transposition defines an entanglement-breaking channel. This map is of course not even positive (so it does not entirely fit with our main considered scenario), but obviously by adding sufficient amount of noise it can be made not only positive but also completely positive. As we show next, the structural approximation to partial transposition does indeed define an entanglement-breaking channel whenever $d_A \geq d_B$, which includes the most relevant case of equal dimension $d_A = d_B$.

This implies that the entanglement detection scheme of Fig. 3.1.a can just be replaced by a sequence of single-copy measurements, see Fig. 3.1.b, being the measurement the one associated to the Holevo form of the entanglement-breaking channel. This alternative scheme is much simpler from an implementation point of view since it does not require any collective measurement, though the measurements are not projective. Moreover, it can never be worse than the previous method, and most likely is better (see also [ASta08]).

3.4.1 Structural Approximations to $1 \otimes T$

Let us then consider the structural approximation to transposition extended to some arbitrary auxiliary space: $1_A \otimes T_B$ [HE02]. Note that, unlike the previous cases, the initial Hilbert space describing the system is now explicitly a product $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \cong \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$. Moreover, generally $1 \otimes \Lambda$ is not the same as $\tilde{1} \otimes \Lambda$, although $1 \otimes \Lambda(\Phi_A^+) = 1 \otimes \Lambda(\Phi_A^+)$, so this problem does not reduce to the previous one. Calculating the witness corresponding to $1_{A} \otimes T_{B}$ one obtains:

$$\tilde{W}_{1_{A} \otimes T} = \frac{p}{(d_A d_B)^2} 1_{AA'} \otimes 1_{BB'} + \frac{1-p}{d_B} \Phi_{A}^{AA'} \otimes F_{BB'},$$

(3.36)
Figure 3.1: The original scheme for direct entanglement detection proposed in [HE02] is shown in (a). Given $N$ copies of an unknown state $\varrho$, it consists of, first, the structural approximation of partial transposition acting on the initial state, followed by optimal estimation of the minimal eigenvalue of the resulting state. In the new scheme, all this structure is replaced by single-copy measurements on the state. The minimal eigenvalue should then be directly estimated from the obtained outcomes.

The condition for structural approximation, positivity of $\tilde{W}_{1 \otimes T}$, is most easily derived by using the identity $F_{BB'} = \Pi_{BB'} - \Pi_{BB'}^{-1}$, where $\Pi_{BB'}$ is the projector on the symmetric subspace $\text{Sym}(\mathcal{H}_B \otimes \mathcal{H}_B')$, and introducing a projector $Q_{AA'} = 1_{AA'} - \Phi_{AA'}$. Then $\tilde{W}_{1 \otimes T}$ becomes:

$$\tilde{W}_{1 \otimes T} = \frac{p}{(d_A d_B)^2} \Phi_{AA'} \otimes \Pi_{BB'} + \frac{1-p}{d_B} \Phi_{AA'} \otimes \Pi_{BB'}^{-1} + \frac{p}{(d_A d_B)^2} \Phi_{AA'} \otimes \Pi_{BB'}^{-1} \left[ Q_{AA'} \otimes \Pi_{BB'} + Q_{AA'}^{-1} \otimes \Pi_{BB'}^{-1} \right]$$

(3.38)

Since only the last term can be negative, one obtains the following condition for structural approximation:

$$p = \frac{d_A^2 d_B}{d_A^2 d_B + 1}.$$  

(3.39)

As a side remark, comparison of the above noise parameter with the one given by Eq. (3.5), for $d = d_B$, shows that $1_A \otimes T_B$ needs more white noise to become completely positive, compared to the amount required by the transposition $T$ alone. In other words, $1_A \otimes \tilde{T}_B$ is less noisy than $1_A \otimes T_B$. 

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3.4. ENTANGLEMENT DETECTION VIA STRUCTURAL APPROXIMATIONS

We proceed to study the separability of \( W_{1\otimes T} \). We begin by finding the partial transposition of \( \tilde{W}_{1\otimes T} \) with respect to the subsystem \( A' \mathcal{T}' \):

\[
\tilde{W}^{T}_{1\otimes T} = \frac{p}{(d_Ad_B)^2} \mathbb{I} + \frac{1-p}{d_A} \Phi^{A' \otimes B'}. \tag{3.40}
\]

Applying the same technique as above (cf. Eq. (3.38)), we find that \( \tilde{W}^{T}_{1\otimes T} \geq 0 \) if and only if:

\[
p \geq \frac{d_Ad_B^2}{d_Ad_B^2 + 1}. \tag{3.41}
\]

Comparing this to the threshold for positivity (3.39), we see that for \( d_A < d_B \), i.e. when the extension is by a space of smaller dimension, there is a gap between positivity and PPT. Hence, in this case, for

\[
\frac{d_A^2}{d_A^2 + 1} \leq p \leq \frac{d_A}{d_A + 1} \tag{3.42}
\]

the state (3.38) is not separable and the map \( \Phi_{A \otimes T_B} \) is not entanglement-breaking in this region. Recall however that this does not represent any counterexample to the conjecture as the initial map is not even positive.

In the case \( d_A \geq d_B \), we will use symmetry arguments to prove the separability of \( W_{1\otimes T} \). From Eq. (3.36) it follows that this state is \( U \mathcal{U} \mathcal{V} \mathcal{V} \)-invariant, where \( U \in U(d_A), V \in U(d_B) \) (cf. Refs. [VW01], [LMD08] where \( U \mathcal{U} \mathcal{V} \mathcal{V} \)-invariant states were studied). Since both groups \( U(d_A) \) and \( U(d_B) \) act independently it is easy to convince oneself [VW01] that the space of \( U \mathcal{U} \mathcal{V} \mathcal{V} \)-invariant operators is spanned by \( \{ \mathbb{I} \otimes \mathbb{I}, \mathbb{1} \otimes \mathcal{F}, \Phi_+ \otimes \mathbb{1}, \Phi_+ \otimes \mathcal{F} \} \). Following the same approach as in subsection 3.2.1, we prove the separability of \( W_{1\otimes T} \) in the partition \( AB : A' \mathcal{T}' \) by showing that the state can be written as convex sum of product states, i.e., it has the following representation

\[
\int d UdV (U_AV_B \tilde{U}_{A'}V_{B'}) \sigma(U_AV_B \tilde{U}_{A'}V_{B'})^\dagger \tag{3.43}
\]

(we omit tensor product signs here for brevity) for some \( \sigma \) separable in the partition \( AB : A' \mathcal{T}' \). Given that states with this invariance are completely described by parameters \( \langle 1 \otimes \mathcal{F} \rangle, \langle \Phi_+ \otimes \mathbb{1} \rangle \) and \( \langle \Phi_+ \otimes \mathcal{F} \rangle \), \( \sigma \) must obey the conditions: \( \text{tr}(\sigma 1 \otimes \mathcal{F}) = \text{tr}(\tilde{W}_{1\otimes T}1 \otimes \mathcal{F}), \text{tr}(\sigma \Phi_+ \otimes \mathbb{1}) = \text{tr}(\tilde{W}_{1\otimes T} \Phi_+ \otimes \mathbb{1}) \) and \( \text{tr}(\sigma \Phi_+ \otimes \mathcal{F}) = \text{tr}(\tilde{W}_{1\otimes T} \Phi_+ \otimes \mathcal{F}) \). Such state \( \sigma \equiv |\varphi \rangle \langle \varphi | \) can be written as

\[
|\varphi \rangle \equiv |\varphi \rangle_{AB} \otimes |\psi \rangle_{A' \mathcal{T}'}
= (\sqrt{\alpha_{00}}|00\rangle + \sqrt{\alpha_{01}}|01\rangle + \sqrt{\alpha_{11}}|11\rangle)|00\rangle \tag{3.44}
\]

for

\[
\begin{align*}
\alpha_{00} &= \frac{d_B}{d_Ad_B^2 + 1} (1 + d_A) \tag{3.45} \\
\alpha_{01} &= \frac{1}{d_Ad_B^2 + 1} (d_A^2 + d_A - d_B(1 + d_A)) \tag{3.46} \\
\alpha_{11} &= 1 - \frac{1}{d_Ad_B^2 + 1} (d_B^2 + d_A). \tag{3.47}
\end{align*}
\]
CHAPTER 3. STRUCTURAL APPROXIMATIONS TO POSITIVE MAPS AND ENTANGLEMENT BREAKING CHANNELS

Notice that, as expected, \( \sigma \) is only well-defined for \( d_A \geq d_B \). According to Eq. (2.18), the map \( \mathbb{1} \otimes \overline{T}(\varrho) \) can be written as

\[
\mathbb{1} \otimes \overline{T}(\varrho) = \int dUdV |w_{UV}\rangle \langle w_{UV}| \text{tr}(d_A d_B |w_{UV}\rangle \langle w_{UV}| \varrho).
\] (3.48)

where \( |v_{UV}\rangle = U \otimes V |\phi\rangle \) and \( |w_{UV}\rangle = U \otimes \overline{V} |\psi\rangle \). Recall also that the integrals over the unitary group defining each depolarization protocol can be replaced by the finite sums of, e.g., Ref. [DCLB00].

In the case \( d_A = d_B = d \), we encounter the structural approximation to the transposition map analyzed in [HE02]. As mentioned, by providing the representation (3.48) we are able to replace the former entanglement detection scheme [HE02] by a much less resource-demanding one. In the original proposal, \( n \) copies of \( \mathbb{1} \otimes \overline{T}(\varrho) \) are prepared, followed by optimal estimation of its minimal eigenvalue by means of a collective projective measurement on the \( n \)-copy state. Now, one should perform local generalized measurements in the \( n \) copies of \( \varrho \) with operators defined in (3.48), which actually might require handling local ancillas of dimension similar to the preparation of \( \mathbb{1} \otimes \overline{T}(\varrho) \). However, the result of our measurements directly provides the estimate the lowest eigenvalue of \( \mathbb{1} \otimes \overline{T} \), with which we avoid the joint measurement of \( N \) copies of the transformed state and significantly reduce the complexity of the previous protocol (see Fig. 3.1).

3.5 Conclusions

In this work, we have studied the implementation of structural approximations to positive maps via measurement and state-preparation protocols. Our findings suggest an intriguing connection between these two concepts, that we have summarized by conjecturing that the structural physical approximation of an optimal positive map defines an entanglement-breaking channel. We prove that the conjecture is true for decomposable maps. Of course, the main open question is now (dis)proving it for the case of nondecomposable maps. We have also applied the same ideas to the study of physical approximations to partial transposition, which is not a positive map, and discuss the implications of our results for entanglement detection.

We would like to conclude this work by giving a geometrical representation of our findings (that should be interpreted in an approximate way). It is well known that the set of quantum states is convex and includes the set of separable states, which is also convex, see also Fig. 3.2. These two sets are contained in the set of Hermitian operators that are positive on product states, which is again convex. Entanglement witnesses belong to this set. If the conjecture is true, it means that the set of optimal witnesses lives in a region which is “opposite” to the set of separable states, in the sense that when mixed with the maximally mixed noise, they enter the set of physical states via the separability region. In addition, we also know that some non-optimal entanglement witnesses cross the entangled region when mixed with white noise. In case the conjecture is true, this provides a peculiar geometric property: none of entanglement witnesses

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4We have seen in Sec. 2.2 that simulating a quantum channel or general measurements on a given quantum state might require auxiliary Hilbert spaces of the same dimension.
3.5. CONCLUSIONS

which lie in the border of the set $W$, on the side where non-optimal witnesses do not have entanglement-breaking SPAs, can be optimal (see Fig.3.2).

Finally, let us mention some further open questions. It would be interesting to extend our studies and ask which classes of positive maps have structural approximation that correspond to partially breaking channels (for definition see [CK06]).
Chapter 4

Noise-robustness of quantum nonlocality

We study the nonlocal properties of states resulting from the mixture of an arbitrary entangled state $\rho$ of two $d$-dimensional systems and completely depolarized noise. The mixing probability $p$ denotes the fraction of entangled state in mixture, and therefore is directly related to the amount of noise affecting the entangled state. We first construct a local model for the case in which $\rho$ is maximally entangled, valid for noise parameter $p$ at or below a certain bound. We then extend the model to arbitrary $\rho$. Our results provide bounds on the resistance to noise of the nonlocal correlations of entangled states. For projective measurements, the critical value of the noise parameter $p$ for which the state becomes local is at least asymptotically $\log(d)$ larger than the critical value for separability.

4.1 Introduction

In 1964, Bell showed that some entangled states are nonlocal, in the sense that measurements on them yield outcome correlations that cannot be reproduced by a local model [Bel64]. All pure entangled states violate such an inequality, hence are nonlocal [Gis91, PR92]. For noisy states, the picture is much subtler. Werner constructed a family of bipartite mixed states which, while being entangled, return outcome correlations under projective measurements that can be described by a local model [Wer89]. This result has been extended to general measurements [Bar02] and more parties [TA06]. Thus, while entanglement is necessary for a state to be nonlocal, in the case of mixed states it is not sufficient (see Sec. 2.3.2 for an overview on the relation between entanglement and nonlocality).

Beyond these exploratory results, little is known about the relation between noise, entanglement, and quantum nonlocality. Understanding this relation, apart from its fundamental interest, is important from the perspective of Quantum Information Science. In this context, entanglement is commonly viewed as a useful resource for various information-processing tasks. However, for some
tasks\textsuperscript{1}, entangled states are useful only to the extent that they exhibit nonlocal correlations. Indeed, in these scenarios two (or more) distant observers, Alice and Bob, directly exploit the quantum correlations (2.22)

\[ P_Q(ab|xy, \rho) = \text{tr}(\rho M^a_x \otimes M^b_y) \tag{4.1} \]

obtained by performing measurements labeled by \(x\) and \(y\) on a distributed entangled state \(\rho\). Recall that if the quantum state \(\rho\) can be simulated by a local model, these correlations can be written as (2.21)

\[ P_L(ab|xy, \rho) = \int d\lambda \omega(\lambda)P^A(a|x, \lambda)P^B(b|y, \lambda), \tag{4.2} \]

where \(\lambda\) denotes a shared classical variable distributed with probability measure \(\omega\), and \(P^A(a|x, \lambda)\) and \(P^B(b|y, \lambda)\) are the local response functions of Alice and Bob (for convenience, we will drop the superscripts in the remaining of the chapter). For all practical purposes then, although \(\rho\) is entangled, it can be replaced by classical correlations and so does not provide any improvement over what is achievable using classical resources.

In this chapter, we estimate the resistance to noise of the nonlocal correlations of bipartite entangled states in \(\mathbb{C}^d \otimes \mathbb{C}^d\), where \(d\) is the local Hilbert space dimension of each subspace. To do this, we analyze the nonlocal properties of states resulting from the mixture of an arbitrary state \(\rho\) with completely depolarized noise,

\[ \rho(p) = p\rho + (1-p)\mathbb{I}/d^2. \tag{4.3} \]

Our goal is to find the minimum amount of noise that destroys the nonlocal correlations of any state \(\rho\), i.e., the maximal value \(p_L\) such that \(\rho(p)\) is local for any \(\rho\) when \(p \leq p_L\). Clearly, for sufficiently small values of \(p \leq p_S\), the state \(\rho(p)\) becomes separable for any \(\rho\), thus local. Here we obtain lower bounds on \(p_L\) that are more constraining than the ones obtained from the separability condition [idZHSL98, VT99, BCJ+99, GB02]. In fact, if we restrict Alice and Bob to perform projective measurements, the bound that we obtain for the locality limit is asymptotically \(\log(d)\) larger than the separability limit.

A key step in the proof of our results is the construction of a local model for states of the form (4.3) when \(\rho = \Phi_d^+\) is the projector on the maximally entangled state,

\[ |\phi_d^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle. \tag{4.4} \]

Thus we also provide a lower bound on \(p_L^\Phi\), defined as the maximal value of \(p\) such that

\[ p_L = p|\phi_d^+\rangle\langle\phi_d^+| + (1-p)\mathbb{I}/d^2 \tag{4.5} \]

is local. This last result implies in particular the existence of entangled states whose nonlocal correlations are more robust to depolarized noise than those of maximally entangled ones.

The results presented here concern both the cases in which Alice and Bob are restricted to projective measurements and have access to generalized measurements. However, we will mainly focus on the first case, due to the similarity

\textsuperscript{1}See Motivation.
between derivations. All relevant information on the extension of our results to general measurements are provided at the end of the chapter. Our results also provide bounds for the notion of state steerability [WJD07].

4.2 Local model for isotropic states

As mentioned, we start by analyzing the case in which the state \( \rho \) in (4.3) is maximally entangled, i.e., we consider the isotropic states (4.5). This family of bipartite states is defined by its invariance under \( \bar{U} \otimes U \) transformations,

\[
\rho_I = \int dU(\bar{U} \otimes U) \rho(U^T \otimes U^\dagger),
\]

(4.6)

for all unitary operators \( U \) on \( \mathbb{C}^d \), and was characterized in [HH99]. Notice that the invariance group of isotropic states is obtained from Werner states (2.29) invariance simply by transposition on one of the subsystems. In the case of \( d = 2 \), both families are the same up to local unitaries, so they share exactly the same properties in terms of entanglement and nonlocality. Observe, however, that for \( d > 2 \), the isotropic states have the interesting feature of being the mixture of maximally entangled state with depolarized noise. That characteristic will turn out essential for the construction of a local model for mixtures of any states with noise.

When Alice and Bob perform local measurements \( M^a_x \otimes M^b_y \) on isotropic states, they obtain the outcome correlations

\[
P_Q(ab|xy) = \frac{p}{d^2} \text{tr}((M^a_x)^T M^b_y) + \frac{1-p}{d^2} \text{tr}(M^a_x) \text{tr}(M^b_y),
\]

(4.7)

which we verify to be very similar to the correlations arising from Werner states (2.30). If we consider von Neumann measurements, the previous equation is further simplified into

\[
P_Q(ab|xy) = \frac{p}{d^2} \text{tr}((\Pi^a_x)^T \Pi^b_y) + \frac{1-p}{d^2}.
\]

(4.8)

Then, our first aim is to construct a local model for isotropic states, that is, to write the quantum probabilities (4.8) in the form (4.2) for some value of the noise parameter \( p \). Our construction is inspired by the model given in [Wer89] for Werner states\(^2\), which are \( U \otimes U \) invariant, and which we adapt to the \( U \otimes \bar{U} \) symmetry of isotropic states.

The local classical variables \( \lambda \) in our model are complex normalized \( d \)-dimensional vectors which we can formally identify with \( d \)-dimensional quantum states \( |\lambda\rangle \). The probability measure \( \omega \) is again induced by the Haar measure on unitary operators and the response functions follow the quantum-like property (2.32). In analogy with the quantum formalism, Alice’s response function is defined as

\[
P(a|x, \lambda) = \langle \lambda | (\Pi^a_x)^T |\lambda\rangle,
\]

(4.9)

where the transposition of Alice’s measurement operator comes from the invariance properties of isotropic states. Following Werner (see [Wer89] and Sec. 2.3.2

\(^2\)In the case of generalized measurements, our local model is based on Barrett’s model for Werner states [Bar02].
one can show this choice restricts Bob’s response function to be associated to a unique positive operator $\hat{P}(b|y)$ such that

$$\int d\omega(\lambda) P(a|x,\lambda)P(b|y,\lambda) = \text{tr} \left( (\Pi^a_x)^T \hat{P}(\Pi^b_y) \right) \quad (4.10)$$

Then, in order for the local model to reproduce the quantum prediction (4.8) we must have

$$\hat{P}(\Pi^b_y) = (p^\phi/d) \Pi^b_y + (1 - p^\phi)/d^2 \mathbf{1} . \quad (4.11)$$

At this point, we want a response function for Bob that meets all the requirements and maximizes the value of $p^\phi$ for which the quantum distribution (4.8) is reproduced by (4.10). This suggests the choice

$$P(b|y,\lambda) = \begin{cases} 1 & \text{if } \langle \lambda|\Pi^b_y|\lambda \rangle = \max_i \langle \lambda|\Pi^b_i|\lambda \rangle \\ 0 & \text{otherwise} \end{cases} . \quad (4.12)$$

which captures the perfect correlations of maximally entangled states, in a similar way to what happened in Werner’s model. To determine the value of $p^\phi$ for which the model holds, it is sufficient to solve (4.10) for the choices for Alice’s (4.9) and Bob’s (4.12) response functions, in the simplest case where $(\Pi^a_x)^T = \Pi^b_y$:

$$p^\phi = \frac{1}{d-1} \left( -1 + d^2 \int \mu(d\lambda) \langle \lambda|\Pi^b_y|\lambda \rangle P(b|y,\lambda) \right) . \quad (4.13)$$

The integral of the last equation can be obtained analytically, after patient algebra, and reads

$$p^\phi = \frac{1}{d-1} \left( -1 + \sum_{k=1}^d \frac{1}{k} \right) \xrightarrow{\text{large } d} \frac{\log(d)}{d} . \quad (4.14)$$

For $d = 2$, $p^\phi = 1/2$ is equal to the critical value for two-dimensional Werner states, as expected since Werner and isotropic states are equivalent up to local unitary transformations. In the limit of large $d$, $p^\phi$ is asymptotically $\log(d)$ larger than the critical probability $p^S_\phi = 1/(d+1)$ for the separability of isotropic states [HH99].

4.3 Local model for mixtures of a quantum state with noise

Our next goal is to generalize the local model for isotropic states to mixed states of the form

$$\rho = p \ketbra{\psi}{\psi} + (1-p) \frac{1}{d^2} , \quad (4.15)$$

where $\ket{\psi}$ is an arbitrary pure state in $\mathbb{C}^d \otimes \mathbb{C}^d$. This automatically also implies a model for the generic states (4.3), since any mixed state $\rho$ is a convex combination of pure states. To do this, we incorporate Nielsen’s protocol [Nie99] for the conversion of bipartite pure states by local operations and classical communication (LOCC) into our model. Recall that a maximally entangled state $\ket{\phi^+_d}$
4.3. LOCAL MODEL FOR MIXTURES OF A QUANTUM STATE WITH NOISE

can be transformed by LOCC in a deterministic way into an arbitrary state $|\psi\rangle$ by a single measurement on Alice’s system, followed by a unitary operation on Bob’s system, depending on Alice’s measurement outcome. Indeed, consider an arbitrary pure entangled state written in its Schmidt form

$$|\psi\rangle = \sum_{j=0}^{d-1} \nu_j |jj\rangle,$$

(4.16)

and denote by $D_\nu$ the $d \times d$ diagonal matrix with entries $(D_\nu)_{jj} = \nu_j$. Taking the $d$ cyclic permutations

$$\pi_i = \sum_{j=0}^{d-1} |j\rangle\langle j+i \text{ (mod } d)|,$$

(4.17)

where $i = 0, \ldots, d-1$, it is possible to write

$$|\psi\rangle = \sqrt{d} (A_i \otimes \pi_i) |\phi_d\rangle$$

for all $i = 0, \ldots, d-1$,

(4.18)

with $A_i = D_\nu \pi_i$. The operators $W_i = A_i^T A_i$ define a measurement, since they are positive and sum to the identity: $\sum_i W_i = I$. In order to convert $|\phi_d\rangle$ into $|\psi\rangle$, Alice first carries out this measurement, obtaining the outcome $i$ with probability $q_i(\lambda) = \langle \lambda | A_i^T A_i^* |\phi_d\rangle$. She then communicates her result to Bob, who applies the corresponding unitary operation $\pi_i$, the resulting normalized state being $|\psi\rangle$, as implied by (4.18).

4.3.1 Mixtures with biased noise

The quantum-like properties of our local model, i.e., the fact that the hidden variable $|\lambda\rangle$ can be thought as a quantum state and the quantum form of the response function (4.9), allows us to adapt Nielsen’s construction to it. Notice that a local model can always be understood in the following operational way. Before the experiment is performed, there is a source that prepares the local hidden-variable $\lambda$, and sends a copy of it to each observer. Alice and Bob will then use these classical correlations (which we can see as instructions from the source) together with their local response functions to decide their output.

Now the idea is that, at the source, the hidden-variable $\lambda$ is pre-processed according to Nielsen’s protocol. A measurement defined by the operators $A_i^*$ is simulated on $|\lambda\rangle$, giving outcome $i$ with probability

$$q_i(\lambda) = \langle \lambda | A_i^T A_i^* |\lambda\rangle.$$

(4.19)

The classical description of the normalized hidden states

$$|\lambda^A_i\rangle = A_i^* |\lambda\rangle / \sqrt{q_i}$$

(4.20)

and

$$|\lambda^B_i\rangle = \pi_i |\lambda\rangle$$

(4.21)

is then sent, respectively, to Alice and Bob, who use them in the response functions (4.9) and (4.12) instead of $|\lambda\rangle$. The joint probabilities $P_L(ab|xy)$
predicted by the model for measurements \( \Pi^a \) and \( \Pi^b \) are thus given by

\[
\int d\lambda \omega(\lambda) \sum_{i=0}^{d-1} q_i(\lambda) P(a|x,\lambda^A) P(b|y,\lambda^B) \]

\[
= \sum_{i=0}^{d-1} \int d\lambda \omega(\lambda) \langle \lambda| A_i^T (\Pi^a_z)^T A_i^* \rangle P(b|\pi^b_i y,\lambda) \quad (4.22)
\]

where we used the property (2.32) of the response functions. At this point, we are interested in knowing which exact quantum state \( \tilde{\rho} \) provides the outcome distribution (4.22). For that, we use (4.10) and (4.11) to write the local distribution (4.22) in the quantum-like form

\[
P_L(ab|xy,\tilde{\rho}) = \sum_{i=0}^{d-1} \left( \frac{p}{d} \text{tr}(A_i^T (\Pi^a_z)^T A_i^* \pi^b_i \pi^a_i) + \frac{1-p}{d^2} \text{tr}(A_i^T (\Pi^a_z)^T A_i^*) \right).
\]

(4.23)

Using Eq. (4.18) and the fact that \( \sum A_i A_i^\dagger = d\sigma \), where \( \sigma = \text{tr}|\psi\rangle \langle \psi| \), one can check that these correlations correspond to the quantum probabilities

\[
\text{tr}(\tilde{\rho} \Pi^a_i \otimes \Pi^b_i)
\]

for the state

\[
\tilde{\rho} = p^\phi |\psi\rangle \langle \psi| + (1-p^\phi)\sigma \otimes \frac{\mathbb{1}}{d}.
\]

(4.25)

Not surprisingly, the measurement at the source modifies the local noise of Alice, which is no longer completely depolarized, and introduces a bias depending on \(|\psi\rangle\).

This result can already be interpreted as a measure of the robustness of the nonlocal correlations of an arbitrary entangled state \(|\psi\rangle\). By mixing a state-dependent local noise, with mixing probability \( 1-p^\phi \), it is possible to wash out the nonlocal correlations of the state \(|\psi\rangle\).

4.3.2 Mixtures with depolarized noise

In order to extend this result to the case of completely depolarized noise, one can add some extra local noise to Alice such that the resulting state has the form (4.15), with the penalty that \( p < p^\phi \). Writing the reduced density matrix \( \sigma \) in its diagonal form \( \sigma = \sum_j \mu_j |j\rangle \langle j| \), and defining

\[
\sigma_k = \sum_j \mu_j^2 |j+k \mod d\rangle \langle j|,
\]

(4.26)

it is clear that the state

\[
q\tilde{\rho} + \frac{1-q}{d-1} \sum_{k=1}^{d-1} \sigma_k \otimes \frac{\mathbb{1}}{d}
\]

(4.27)

has the form (4.15) for \( q(1-p_d) = (1-q)/(d-1) \), in which case the probability \( p \) is given by

\[
p^\phi = \frac{p^\phi}{(1-p^\phi)(d-1)+1} \xrightarrow{\text{large } d} \frac{\log(d)}{d^2}.
\]

(4.28)
4.4. BOUNDS ON NOISE ROBUSTNESS OF NONLOCALITY

| Separability | $|\phi_d\rangle$ | arbitrary $\rho$ |
|--------------|-----------------|------------------|
| $p_S^\phi = \frac{1}{d+1}$ | $\frac{1}{d+1} \leq ps \leq \frac{2}{d^2+2}$ |
| **Locality** (Proj. Meas.) | $\Theta\left(\frac{\log d}{d}\right) \leq p_L^\phi \leq \frac{\pi^2}{16K} \simeq 0.67$ | $\Theta\left(\frac{\log d}{d}\right) \leq p_L \leq \Theta\left(\frac{4}{(\sqrt{2}-1)d}\right)$ |
| **Locality** (Gen. Meas.) | $\Theta\left(\frac{3}{e^2d}\right) \leq p_L^\phi \leq \frac{\pi^2}{16K} \simeq 0.67$ | $\Theta\left(\frac{3}{e^2d}\right) \leq p_L \leq \Theta\left(\frac{4}{(\sqrt{2}-1)d}\right)$ |

Table 4.1: Asymptotic bounds on the critical noise threshold for separability ($p_S$) and locality ($p_L$) for maximally entangled states ($|\phi_d\rangle$) and arbitrary states ($\rho$). For maximally entangled states, $p_S^\phi$ is given in [HH99]; the lower bounds for $p_L^\phi$ follow from eqs. (4.14) and (4.35); and the upper bounds from [CGL*02], where $K$ is Catalan’s constant. For arbitrary states, bounds for $p_S$ were derived in [GB02]; the lower bounds for $p_L$ are obtained from those for the maximally entangled states using eq. (4.28); the upper-bounds are those of [ADGL02].

The state (4.27) is clearly local, since it is a convex combination of local states.

We have thus shown that the noisy states (4.3) have a local model for projective measurements whenever $p \leq p^\rho$. The probabilities $p^\rho$ and $p^\varphi$ represent the main results of this work and provide lower bounds on $p_L^\rho$ and $p_L$. Several implications of our findings are discussed in what follows.

4.4 Bounds on noise robustness of nonlocality

First of all, one may ask about the tightness of our bound on $p_L^\varphi$. Actually, our model is based on Werner’s construction, and this model is known not to be tight in the case $d = 2$, for which correlations are local up to $p = 0.66$ [AGT06]. Even though not tight, it would be interesting to understand whether the model predicts the right asymptotic dependence with the Hilbert space dimension $d$.

An upper bound on $p_L$ follows from the results of [ADGL02], where it was shown that a state of the form

$$\rho_2 = p|\phi_2\rangle\langle\phi_2| + (1 - p)I/d^2,$$

(4.29)

where $|\phi_2\rangle = 1/\sqrt{2}(|00\rangle + |11\rangle)$ is a projector onto a two-qubit maximally entangled state, violates the Clauser-Horne-Shimony-Holt inequality [CHSH69] whenever $p > p^\varphi$, where

$$p^\varphi = \frac{4(d-1)}{(\sqrt{2}-1)d^2 + 4d - 4} \xrightarrow{\text{large } d} \frac{4}{(\sqrt{2}-1)d},$$

(4.30)

which tends to zero when $d \to \infty$. This result together with our previous model thus implies that

$$p^\varphi \leq p_L \leq p^\rho. $$

(4.31)

Our results, when combined with (4.30), also provide a strict proof of the fact that the nonlocal correlations of maximally entangled states, for projective measurements, are not the most robust ones under addition of depolarized noise.
Indeed, we have a local model for isotropic states whenever \( p \leq p^φ \), while there exist quantum states of the form (4.3) violating a Bell inequality when \( p > p^ϱ \). For sufficiently large dimension, \( p^{φ^2} < p^φ \) so we have a Bell inequality violation in a range of \( p \) for which we have shown the existence of a local model for isotropic states.

It is also interesting to compare the bounds derived here for nonlocality with those known for entanglement. To our knowledge, the best upper and lower bound on the critical probability \( p_S \) such that the states (4.3) are guaranteed to be separable were obtained in [GB02]:

\[
\frac{1}{d^2 - 1} \leq p_S \leq \frac{2}{d^2 + 2}. \tag{4.32}
\]

Interestingly, the upper bound is obtained (as above), for the case in which the state \( ϱ \) in (4.3) is equal to a projector onto \( |φ⟩^3 \). Comparing with Eq. (4.28), we see that the critical noise probability for nonlocality under projective measurements is, at least, asymptotically \( \log(d) \) larger than the one for separability, as it is for isotropic states.

### 4.5 Extension for generalized measurements

Finally, let us briefly mention how the above results can be extended to the case of general measurements. The idea is, as above, to start by constructing a model for isotropic states, adapting the one for Werner states of [Bar02]. As noted before, it is sufficient to simulate generalized measurements defined by operators \( M^a_x = c^a_x Π^a_x \) and \( M^b_y = c^b_y Π^b_y \) proportional to one-dimensional projectors \( Π^a_x \) and \( Π^b_y \) in order for Alice and Bob to be able to simulate any general measurement. In our corresponding model, the hidden states are again vectors \( |λ⟩ \) in \( C^d \) chosen with the Haar measure \( μ \). Alice’s response function is basically the same as before,

\[
P(a|x, λ) = (λ|M^a_x|^T|λ⟩, \tag{4.33}
\]

while, inspired by [Bar02], Bob’s response is

\[
P(b|y, λ) = (λ|M^b_y|^T|λ⟩ \Theta \left( (λ|Π^b_y|^T|λ⟩ - 1/d \right) + c^b_y \left[ 1 - \sum_k (λ|M^b_k|^T|λ⟩ \Theta \left( (λ|Π^b_k|^T|λ⟩ - 1/d \right) \right), \tag{4.34}
\]

where \( Θ \) is the Heaviside step function. The evaluation of the integral (4.2) with the definitions (4.33) and (4.34) can be done following the same steps as in [Bar02] and yields the joint measurement outcome probabilities for an isotropic state with the critical value

\[
p^φ = \left( \frac{3(d - 1)(d - 1)^{d - 1}}{(d + 1)d^d} \right) \frac{3}{d} \rightarrow \frac{1}{d}. \tag{4.35}
\]

\(^3\)This can be a consequence that our intuition, and therefore the developed methods, is much stronger for the case of singlet states.

\(^4\)See Sec. 2.3.2.
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Since this model has the same quantum-like properties as the one for projective measurements, according to definition (4.33), it can also be extended to arbitrary noisy states (4.3) using Nielsen’s protocol. The corresponding critical probability is given by (4.13) with $p^\phi$ replaced by the above value of $\tilde{p}^\phi$.

In conclusion, we have obtained bounds on the robustness of the nonlocal correlations of arbitrary entangled states. Our results are summarized in Table I. In the particular but interesting case where the state is maximally entangled, we derived better bounds by exploiting the symmetry of isotropic states. Apart from their fundamental significance, our results are interesting from the point of view of the characterization of quantum information resources: if the noise affecting a state is larger than our bounds, its outcome correlations for local measurements can be reproduced by classical means alone by the use of local models presented here.

The local model for isotropic states presented in this chapter was independently derived in [WJD07] in the context of state steerability. We note that all our models imply the non-steerability of the corresponding quantum states because Alice’s response function is always quantum (see [WJD07] for details).
Chapter 5

Multipartite fully-nonlocal quantum states

We present a general method to characterize the quantum correlations obtained after local measurements on multipartite systems. Sufficient conditions for a quantum system to be fully-nonlocal according to a given partition as well as (genuinely) multipartite fully-nonlocal are derived. These conditions allow us to identify all completely-connected graph states as multipartite fully-nonlocal quantum states. Moreover, we show that this feature can also be observed in mixed states: the state composed by five copies of the four-qubit Smolin state (a biseparable and bound entangled state), symmetrically distributed by five parties, is multipartite fully-nonlocal.

5.1 Introduction

Quantum nonlocality is an intrinsic quantum feature and lies behind several applications in quantum information theory. The majority of known results on quantum nonlocality refer to the bipartite scenario and, even though multipartite quantum correlations are a potential valuable resource for multiparty quantum information tasks, their characterization remains a general unsolved problem.

The most common method to detect nonlocal correlations is through the violation of a Bell inequality. It is however unclear whether the amount of violation quantifies nonlocality in a meaningful way [MS07]. Indeed, the amount of violation of a Bell inequality is intrinsically associated to fixed parameters of the experiment, namely the number of allowed measurement settings and outcomes. However, for the bipartite scenario, the EPR-2 formalism for the study of nonlocality (2.52), naturally leads to a quantitative notion of nonlocality [EPR92]1. Recall that the EPR-2 decomposition considers that the obtained joint probability distribution of outcomes, $P_{\rho}$, can be written as a convex sum of a local distribution $P_{L}$ and a nonlocal distribution $P_{NS}$,

$$P_{\rho}(ab|xy) = p_L P_L(ab|xy) + (1 - p_L)P_{NL}(ab|xy).$$

1See Section 2.3.3.
CHAPTER 5. MULTIPARTITE FULLY-NONLOCAL QUANTUM STATES

The nonlocal distribution $P_{NS}(ab|xy)$ is in principle arbitrary, but it has to be no-signaling since $P_\rho(ab|xy)$ and $P_L(ab|xy)$ have this property. For the given state $\rho$, the goal is to identify the decomposition which maximizes the weight of the local part, $p_L$, among all possible local measurements. The solution to this optimization problem, $p_L(\rho)$, is clearly a function of the state only and can be interpreted as a measure of its nonlocal correlations.

In the bipartite case, some of the most basic questions on quantum nonlocality have been answered. In what follows it will be useful to express these findings in terms of the properties of the EPR-2 decomposition (5.1), namely of the weight $p_L$. For instance, it is known that there exist mixed entangled states which are local or, equivalently, have $p_L = 1$ [Wer89, Bar02]. On the other hand, Gisin’s theorem proves that every pure entangled bipartite state violates a Bell inequality, which means that these states have $p_L < 1$ [Gis91]. Moreover, it is known that fully-nonlocal states exist, as maximally entangled states have $p_L = 0$ [EPR92, BKP06].

Moving to the multipartite scenario, we can find parallel results to those on bipartite quantum nonlocality. For instance, there exist genuine multipartite entangled states which are fully local, since their outcome distributions can be modeled by (2.58) [TA06]. On the contrary, every pure entangled multipartite state violates a Bell inequality [PR92]. Also, no fraction of the statistics obtained by measuring any state manifesting a GHZ-like paradox can be described by a completely local model (2.58) [GHSZ90, BKP06]. However, these results say nothing about the presence of genuine multipartite nonlocal correlations, i.e. those nonlocal correlations established between all the $m$-parties of an $m$-partite quantum state. As mentioned before \(^2\), identification of genuine multipartite nonlocality requires the use of Svetlichny inequalities [Sve87, SS02, CGP+02].

In this chapter, we present a general method to study multipartite nonlocality, including genuine multipartite, in the no-signaling scenario. It is based on a multipartite version of the EPR-2 decomposition (5.1) and on the results of measurements held on a subset of the parties sharing a multipartite quantum state. Using our framework to study genuine multipartite nonlocality we provide sufficient conditions to detect genuine multipartite full-nonlocal correlations, which we simply designate by multipartite fully-nonlocal. We are able to identify the completely-connected graph states [HDrE+06] as the first example of multipartite fully-nonlocal states, proving the existence of these states for any number of parties. Furthermore, we prove that multipartite full nonlocality can also be observed in the mixed-state case: the state composed of five copies of the 4-qubit Smolin state [Smo01], symmetrically distributed by five parties, defines a multipartite fully-nonlocal mixed quantum state.

5.2 Characterizing multipartite nonlocality

In order to introduce the complex structure of correlations in a multipartite scenario, we start by considering an extension of the EPR-2 decomposition to

\(^2\)See section 2.3.4
the tripartite case,

\[ P(abc|xyz) = p_L P^{A:B:C} + p_{L:NS} P^{A:BC} + p_{B:AC} P^{C:AB} + p_{C:AB} P^{NS:} + p_{NS:} P^{NS:}, \]  

(5.2)

with \( p_L + p_{L:NS} + p_{B:AC} + p_{C:AB} + p_{NS:} = 1 \). Here, the distribution \( P^{A:B:C} \) is completely local (2.58) and therefore it strictly contains classical correlations among the outcomes of the local measurements. On the other hand, \( P_{L:NS} \) represent local-nonlocal hybrid models\(^3\), for instance,

\[ P^{A:BC}_{L:NS} = \int d\lambda \omega(\lambda) P(a|x,\lambda) P(bc|yz,\lambda). \]  

(5.3)

Note that contrary to what happens in the bipartite case, the distributions appearing in the different nonlocal terms of the EPR-2 decomposition (5.2) are arbitrary and may in principle allow signaling between the corresponding parties. However, here we work in the no-signaling scenario and thus, all the terms appearing in the decomposition are assumed to be compatible with this principle. The intuition is that no object of the theory is allowed to signal\(^4\).

Finally, the component \( P_{NS} \) is the only to contain genuine tripartite nonlocal correlations. This decomposition can easily be extended to an arbitrary number of parties, \( m \). The richness of multipartite correlations expresses itself by the rapid growth of hybrid local-nonlocal terms with \( m \).

Observe how this multipartite version of the EPR-2 decomposition clearly distinguishes bipartite from genuine \( m \)-partite quantum nonlocality. At this point we stress that by the fact that we are considering the completely no-signaling scenario, our results can only be linked to violation of Svetlichny-like inequalities which also consider this same scenario. We will return to this point later. Resuming our discussion, we know that in order for a quantum state to violate a standard Bell inequality it is sufficient that it has \( p_L < 1 \), while it violates a Svetlichny-like (no-signaling) inequality if and only if \( p_{NS} > 0 \).

Analogously, bipartite full nonlocality is present when \( p_L = 0 \) but multipartite (no-signaling) full nonlocality is synonymous of the much stronger condition \( p_{NS} = 1 \). Thus, the parameter \( p_{NS} \) is the relevant quantity when studying genuine multipartite nonlocality. In what follows, we focus our analysis on this quantity and provide a sufficient criterion to detect multipartite fully-nonlocal correlations.

For clarity, let us rephrase some known results on multipartite nonlocality and stress the aim of the present work in terms of the generalized EPR-2 decomposition (5.2). The fact that some genuine entangled states are fully local implies \( p_L = 1 \), while every multipartite pure entangled state violates a Bell inequality [PR92] corresponds to \( p_L < 1 \). As commented before, any state exhibiting a GHZ-like paradox has \( p_L = 0 \) [GHSZ90, BKP06]. Here we will provide a sufficient criterion for a multipartite state to have \( p_{NS} = 1 \). This criterion identifies several multipartite pure quantum states, as well as a mixed one, as multipartite fully-nonlocal.

\(^3\) See section 2.3.4.

\(^4\) One can even argue that from a physical point-of-view, the only reasonable scenario is the one where signaling is completely forbidden. This does not invalidate the original Svetlichny approach, where signaling inside the partition blocks is allowed, which is certainly interesting from a resource point-of-view. If quantum correlations are stronger than these models, it means that they cannot be replaced the shared randomness aided by communication inside the partitions.

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5.3 Criterion to detect full nonlocality in the multipartite scenario

To start, we introduce a different version of multipartite EPR-2 decomposition (5.2), which focuses on the correlations across a specific bipartition of the composite system. Consider an $m$-partite state $\rho$ and a bipartition $A : B$, where $A$ contains $k$ parties and $B$ the remaining $m - k$. To simplify the notation, assume that $A$ contains the first $k$ parties and $B$ the $m - k$ remaining ones. Measurement settings in each partition are labeled by $X = (x_1, \ldots, x_k)$ and $Y = (x_{k+1}, \ldots, x_m)$, and the respective outcomes are $A = (a_1, \ldots, a_k)$ and $B = (a_{k+1}, \ldots, a_m)$. The new version of the multipartite EPR-2 decomposition is then given by

$$P_\rho(AB|XY) = p^{AB}_L P^{AB}_L(AB|XY) + (1 - p^{AB}_L) P^{NS}_{NS}(AB|XY). \quad (5.4)$$

We use the subscript $L$ to indicate locality in the partition $A : B$, although the distribution $P^{AB}_L(AB|XY)$ is hybrid, i.e.

$$P^{AB}_L(AB|XY) \equiv \int d\lambda \omega(\lambda) P(a_1 \cdots a_k|x_1 \cdots x_k, \lambda) P(a_{k+1} \cdots a_m|x_{k+1} \cdots x_m, \lambda), \quad (5.5)$$

and allows any no-signaling correlations among members of the same partition. The nonlocal component $P^{NS}_{NS}(AB|XY)$ in (5.4) contains all correlations not modeled by (5.5). Among all possible decompositions (5.4), we focus on the one maximizing the weight of the local part, $p^{AB}_L$. We then define full nonlocality with respect to the partition $A : B$ by $p^{AB}_L = 0$. It is evident that this multipartite version (5.4) strongly resembles the original bipartite EPR-2 decomposition (5.1): we will see that this generalized bipartite decomposition form is essential in what follows.

Before showing how to detect multipartite fully-nonlocality, we must present a method to identify full-nonlocal correlations across a given bipartition of an $m$-partite quantum state. In fact, the following theorem constitutes the core of our results.

**Theorem 1.** An $m$-partite state $\rho$ is fully-nonlocal across a given bipartition $A : B$ ($p^{AB}_L = 0$), in the no-signaling scenario, if it is possible to create a maximally entangled state between one party in each partition, for all outcomes of suitable local measurements on the remaining parties.

**Proof:** We are interested in showing that the outcome distribution of $\rho$, in the EPR-2 decomposition (5.4), has $p^{AB}_L = 0$. From [BKPO06] we know that any bipartite maximally entangled state is fully nonlocal, i.e., $p_L = 0$. The proof of Theorem 1 will then follow by contradiction: $p^{AB}_L > 0$ would imply $p_L > 0$ for the maximally entangled state.

Indeed, assume there is a positive local weight $p^{AB}_L > 0$. For ease of notation, consider the case in which a maximally entangled state can be created between party $A_1$, belonging to $A$, and $A_m$, belonging to $B$, by local measurements in the remaining parties, $M_2 \otimes \cdots \otimes M_{m-1}$. We label the measurement settings by $\tilde{x}$ and $\tilde{y}$, and their respective outcomes by $\tilde{a}$ and $\tilde{b}$. For every outcome $(\tilde{a}, \tilde{b})$, parties $A_1$ and $A_m$ are projected onto the maximally entangled state $\psi_{2,\tilde{a},\tilde{b}}$ (See
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MULTIPARTITE SCENARIO

Fig. 5.1). Given that we only consider no-signaling distributions, the following identity holds true for every component of the EPR-2 decomposition (5.4):

\[
P(a_1a_m, \tilde{a}\tilde{b}|x_1x_m, \tilde{x}\tilde{y}) = P(a_1a_m|x_1x_m, \tilde{a}\tilde{b}\tilde{x}\tilde{y})P(\tilde{a}\tilde{b}|\tilde{x}\tilde{y}).
\]

(5.6)

This simply expresses the usual no-signaling condition: the outcome of measurements on parties \(A_2 \ldots A_{m-1}\) cannot depend on the choice of measurements by distant parties \(A_1\) and \(A_m\).

Since we assumed that \(P^A_B > 0\), a fraction of the statistics of the state \(\rho\) is modeled by the hybrid model (5.5). Then, for every measurements choice \((\tilde{x}, \tilde{y})\) on the \(N - 2\) parties, there is at least one outcome \((\tilde{a}, \tilde{b})\) which the hybrid model predicts with non-zero probability: \(P^A_B(\tilde{a}\tilde{b}|\tilde{x}\tilde{y}) > 0\). The post-measurement state associated to this outcome, \(\psi^\tilde{a}\tilde{b}\), is maximally entangled by assumption and has correlations according to the induced EPR-2 decomposition

\[
P_{\tilde{a}\tilde{b}}(a_1a_m|x_1x_m) = p_L P^A_B(a_1a_m|x_1x_m, \tilde{a}\tilde{b}\tilde{x}\tilde{y}) + (1 - p_L)P^A_{\tilde{NS}}(a_1a_m|x_1x_m, \tilde{a}\tilde{b}\tilde{x}\tilde{y}),
\]

(5.7)

with local weight

\[
p_L = \frac{P^A_B}{P^A_B(\tilde{a}\tilde{b}|\tilde{x}\tilde{y})} P_\tilde{a}(\tilde{a}|\tilde{x}).
\]

(5.8)

Notice that the induced non-local distribution is well-defined because its only constraint, the no-signaling condition, is not affected\(^5\). Also, the fact that we can apply the no-signaling condition to every component of the hybrid model (5.5), guarantees that we obtain the valid induced hybrid distribution

\[
P_L(a_1b_1|x_1y_1, \tilde{a}\tilde{b}\tilde{x}\tilde{y}) = \int d\lambda \tilde{\omega}(\lambda) P(a_1|x_1, \tilde{a}\tilde{x}\lambda) P(b_1|y_1, \tilde{b}\tilde{y}\lambda),
\]

(5.9)

where the induced probability density is

\[
\tilde{\omega}(\lambda) = \frac{\omega(\lambda)P(\tilde{a}|\tilde{x}, \lambda)P(\tilde{b}|\tilde{y}, \lambda)}{P_L(\tilde{a}\tilde{b}|\tilde{x}\tilde{y})}.
\]

(5.10)

According to (5.8), the local weight \(p_L\) is necessarily positive. This is known to be impossible since it corresponds to a maximally entangled state \(\psi^\tilde{a}\tilde{b}\) \([\text{BKP06}]\).

Therefore, the distribution for the state \(\rho\) must be fully nonlocal on the partition \(A : B\), \(p^A_B = 0\). \(\square\)

Notice that we can relax the assumptions of Theorem 1.: it is sufficient to obtain a bipartite state with \(p_L = 0\) for every outcome. An important remark on the previous result is that the proof is presented as a sequence of measurements only by clarity reasons. In fact, given that all measurements define spatially-separated events, the results of measurements on parties \(A_2 \ldots A_{m-1}\) are guaranteed to be independent from the measurement choices of parties \(A_1\) and \(A_m\) by the no-signaling principle. Therefore, there is no need to impose any time-ordering on the measuring events and we are in the most standard framework of nonlocality, which considers single local measurements in space-like separated systems.

\(^5\)If \(P^A_{\tilde{NS}}(\tilde{a}\tilde{b}|\tilde{x}\tilde{y}) = 0\), the non-local distribution is not well-defined but the induced state \(\psi^\tilde{a}\tilde{b}\) is local and our proof still holds.
We are now ready to finally present the result which, combined with Theorem 1, provides the sufficient criterion to detect multipartite full nonlocality.

**Theorem 2.** A probability distribution is multipartite fully-nonlocal ($p_{NS} = 1$) if and only if it is fully nonlocal ($p_{AB}^{L,B} = 0$) in every bipartition $A : B$.

*Proof:* The proof proceeds again by contradiction. Assume that $p_{NS} < 1$. Then, there is at least one local/hybrid model in the EPR-2 decomposition (5.2) with positive weight. However, there is always a bipartite splitting of the parties such that this term contributes to the corresponding local part in Eq. (5.4). But this is now in contradiction with the fact that $p_{AB}^{L,B} = 0$ for every bipartition. Then, $p_{NS}$ is equal to one for the initial distribution. $\square$

### 5.4 Multipartite fully-nonlocal states

#### 5.4.1 Completely-connected graph states

From Theorems 1 and 2 we can immediately identify the completely-connected graph states as being multipartite fully-nonlocal states. This comes from the fact that these states fulfill all the necessary requirements: for any pair of qubits, there are local Pauli measurements on the remaining $N - 2$ parties which project the pair of particles into a maximally entangled state for every measurement outcome [HDrE*06, HEB04].

Graph states are known to possess several peculiar features like being perfect quantum channels for quantum communication, or (some of them) universal resources for measurement-based quantum computation [HDrE*06]. The fact that completely-connected graph states are multipartite fully-nonlocal is one more interesting feature of this important class of multipartite entangled states.
5.4.2 A multipartite fully-nonlocal mixed state

We now present the first known example of mixed state which contains multipartite fully-nonlocality. This example is based on the Smolin state, a four-partite bound entangled state given by [Smo01]

$$\rho_{ABCD}^S = \frac{1}{4} \sum_{i=0}^{3} |\psi_i\rangle \langle \psi_i|_{AB} \otimes |\psi_i\rangle \langle \psi_i|_{CD},$$  \hspace{1cm} (5.11)

where $|\psi_i\rangle$ denote the four Bell states.

The Smolin state is biseparable in all two-qubit versus two-qubit partitions, and consequently no entanglement can be distilled from it by local operations and classical communication. Interestingly, it was shown in Ref. [SST03] that the combination of five copies of the Smolin state, distributed by five parties according to

$$M^S = \rho_{ABCD}^S \otimes \rho_{ABCE}^S \otimes \rho_{ABDE}^S \otimes \rho_{ACDE}^S \otimes \rho_{BCDE}^S,$$  \hspace{1cm} (5.12)

is distillable. The distillation protocol presented in [SST03] allows obtaining deterministically from $M^S$, a singlet state between any pair of parties. It follows that local operations, aided by classical communication, can always create a maximally entangled state between any partition of the state $M^S$. This nearly matches our conditions for the presence of full multipartite nonlocality, being the only difference the presence of communication. We are in the no-signalling scenario and therefore all communication should be removed $^6$. It happens that using the distillation protocol without communication, enables any pair of parties to obtain, for a given outcome $i$ of the composed measurements in the other parties, a state

$$|\psi_i\rangle = \mathbb{1} \otimes \sigma_i |\psi^-\rangle,$$  \hspace{1cm} (5.13)

where $\sigma_i$ correspond to the Pauli matrices. Since $|\psi_i\rangle$ is one of the four Bell states, a maximally-entangled state is created for all outcomes of local measurements in the other parties, and our sufficient criterion says that $M^S$ has $p_{NS} = 1$. This result proves the existence of mixed states which contain multipartite fully-nonlocal quantum correlations.

5.5 Conclusion

We have seen how generalizations of the EPR-2 decomposition for quantum probability distributions and outcomes of partial measurements on quantum systems can be used to study multipartite nonlocal correlations. Our formalism gives sufficient conditions to detect multipartite full nonlocality and identifies all completely-connected graph states as examples of multipartite fully-nonlocal states. This result solves a fundamental question concerning the characterization of nonlocal quantum correlations: in the no-signalling scenario, multipartite fully-nonlocal states exist for any number of parties. In addition, we also provide an example of such extreme nonlocality for a multipartite mixed state.

$^6$Of course that communication across the partitions is forbidden, as it could create non-locality in that splitting. However, certain patterns of communication inside the partitions are also allowed by our sufficient conditions for multipartite full nonlocality.
Finally, our work opens new questions on the characterization of multipartite nonlocality. Would our conclusions be affected if the different terms appearing in the decomposition were not constrained by the no-signaling principle? Providing this extra resource could give the hybrid models the ability to reproduce a fraction of the quantum probability distribution, and then full nonlocality would be lost. Another interesting open question is to characterize the set of all multipartite fully-nonlocal quantum states. Is the derived sufficient criterion also necessary to identify multipartite fully-nonlocal quantum correlations?
Chapter 6

Activation of quantum nonlocality

In the multipartite scenario, nonlocality of a quantum state is guaranteed whenever measurements on a subset of its parties project the subsystem of the remaining parties in a nonlocal state. We use this fact as a tool to study both bipartite and multipartite nonlocal correlations. By distributing copies of bipartite states over several parties, according to given configurations, we are able to obtain the following results. First, we establish a direct link between one-way distillability and quantum nonlocality. With that we are able to improve the noise threshold for which isotropic states (mixtures of maximally-entangled state with white noise) are known to be nonlocal. Finally, we prove that the nonlocality of quantum states can be activated. There exist quantum states, having no nonlocal correlations when considered individually, but for which local collective measurements on a sufficiently large number of copies exhibit this resource. We provide examples of activation of bipartite and genuine multipartite nonlocal correlations.

6.1 Introduction

Non-additivity seems to be an intrinsic feature of Quantum Information Theory. It is known that the combined use of quantum objects can be more advantageous than the sum of individual uses. Take, for instance, quantum channels and their capacity to transmit quantum, classical and private [SY08, Has09, LWZ09]. Or quantum states and their entanglement properties [Has09, SST03]. It is also clear that entanglement is the quantum property behind all these effects: entangled inputs enhance the capacity of quantum channels, while entangling measurements are necessary for the activation of distillable entanglement. Activation of resources represents the strongest form of non-additivity: quantum objects which are useless for a given task become useful when taken together.

It is a fundamental open question whether similar non-additivity effects can be found in the nonlocal correlations of entangled states: is there a local state $\rho$ such that the state composed of many copies of it, distributed among several parties, violates a Bell inequality? Do collective local measurements on a subset of the parties of multipartite states help the detection of nonlocal-
In this work we provide affirmative answers to both questions: we use Bell tests performed in a multipartite scenario to prove that both bipartite and genuine multipartite quantum nonlocality can be activated. We also show that multipartite measurement strategies can be a simpler alternative to the usual construction of Bell inequalities for the detection of quantum nonlocality.

Our method is based on the fact that a multipartite quantum state is nonlocal if measurements on a subset of its parties project the subsystem of the remaining parties in a nonlocal state. We take copies of bipartite quantum states distributed by several parties, according to specific configurations, and study the nonlocal properties of the constructed state. Following this approach, we start by providing a direct link between nonlocality and one-way entanglement distillability, i.e. the possibility of extracting entanglement from several copies of a state by measurements in only one party. Consequently, we are able to improve the noise threshold for which the well-studied family of isotropic states, for large dimension $d$, are known to violate a Bell inequality. Second, we show examples of quantum states whose nonlocality can be revealed by collective local measurements. More precisely, we show (i) a local state $\rho_{AB}$ such that $\rho_{AB_1} \otimes \rho_{AB_2} \otimes \ldots \otimes \rho_{AB_M}$ is nonlocal for large $M$, where $B_i$ represents different parties; and (ii) a state $\rho_{ABC}$ without multipartite genuine correlations, such that $\rho_{MABC}^{\otimes M}$ becomes fully (genuine) multipartite nonlocal.

The entangled states used in our examples are either the maximally entangled state
\[
|\phi^+_d\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle ,
\]
(6.1)
or its mixture with depolarized noise, the well-known isotropic states,
\[
\rho_{MABC}^{\otimes M} = p |\phi^+_d\rangle \langle \phi^+_d| + (1-p) \frac{I}{d^2} ,
\]
(6.2)
where $p$ is the noise parameter. We already know from chapter 4 that, although apparently simple in its form, there is still no complete understanding of the nonlocality properties of this family of quantum states (see fig.6.1). In particular, there is a gap between the value of $p$ below which the states are local and the minimum value of $p$ necessary for them to violate of any known Bell inequality. Consider first the simplest case of $d = 2$. The isotropic states are local under projective measurements for $p \leq 0.66$ [AGT06] and local under general measurements for $p \leq 5/12$ [Bar02]. On the other hand, they violate a Bell inequality for $p \gtrsim 0.705$ [V08]. For many years, the lowest bound was $p \gtrsim 1/\sqrt{2} \approx 0.707$, obtained by the violation of the CHSH inequality [CHSH99]. Only recently it was shown not to be optimal: using a Bell inequality with 465 (!) measurement settings per party, it was possible to improve this bound by $\sim 0.1\%$. That is indeed a good indicator on how difficult it might be to detect nonlocality. For higher dimensions, we have seen that there is a local model for any projective measurement experiment for $p \leq O(\log d/d)$ and (6.2) violates a Bell inequality for $p \gtrsim 0.67$ in the limit of large $d$ [CGL+02].

### 6.2 Nonlocality revealed by multipartite strategy

The multipartite measurement strategy we use to obtain our results is basically the same that enabled generalizing Gisin’s theorem to the multipartite case...
6.2. NONLOCALITY REVEALED BY Multipartite STRATEGY

Figure 6.1: Nonlocality of isotropic states. Upper figure. For $d = 2$, the isotropic states (6.2) violate the CHSH inequality [CHSH69] for $p > 1/\sqrt{2} \approx 0.707$ and violate Bell inequality composed by 465 settings per party for $p \gtrsim 0.705$ [VÕ8]. A local model for projective measurements is possible for $p \lesssim 0.66$ [AGT06]. This limit can also be shown to hold if general dichotomic measurements are considered (see text and Appendix 2). Note also that this bound drops down to $p \leq 5/12$ if general measurements are considered [Bar02]. Here we present a multipartite strategy that shows the nonlocality of $\rho^{(d)}$ for $p \gtrsim 0.64$, when performing collective measurements on many copies of the state. This proves the activation of bipartite nonlocality, considering projective measurements only. Lower figure. For high dimensions, the isotropic states violate the CGLMP inequality for $d \gtrsim 0.67$ (in the asymptotic limit $d \to \infty$). This bound is significantly improved to $p = 1/2$ using the multipartite strategy described in Fig. 6.3.
There we have seen that, for every $N$-partite pure entangled state, there exists a projection on $N-2$ parties that leave the remaining parties in a pure bipartite entangled state. Since these always violate a CHSH inequality, we could derive an $N$-partite Bell inequality violated by the original multipartite state. Evidently, this is a special case of the following.

**Observation 1.** Consider $N$ spacelike separated quantum systems described by the state $\rho_{A_1...A_N}$. Suppose that in a group of $N-k$ parties, each one performs a local measurement on their respective systems. If, for a given outcome of these measurements, the remaining $k$ parties are left in a nonlocal state, the original state $\rho_{A_1...A_N}$ must be nonlocal.

For completeness, we can repeat the argument used before, in order to prove Observation 1. Consider that set of local measurements $M^{\tilde{a}_{k+1}}_{A_{k+1}} \otimes ... \otimes M^{\tilde{a}_N}_{A_N}$ is performed on parties $A_{k+1}...A_N$, where $\tilde{y} = x_{k+1}...x_N$ labels the choice of measurement settings and $\tilde{b} = a_{k+1}...a_N$ the correspondent outcomes. If the remaining state $\rho_{\tilde{a},\tilde{y}}$ is nonlocal, it must violate some Bell inequality

$$\sum_{a_1...a_k,x_1...x_k} c_{a_1...a_k,x_1...x_k} P(a_1...a_k|x_1...x_k,\tilde{b}\tilde{y}) \leq k. \tag{6.3}$$

Since the local measurements define spacelike separated events, the no-signaling condition guarantees

$$P(a_1...a_k\tilde{b}|x_1...x_k\tilde{y}) = P(a_1...a_k|x_1...x_k,\tilde{b}\tilde{y})P(\tilde{b}\tilde{y}). \tag{6.4}$$

Then, multiplying both sides of (6.3) by the probability of the event $(\tilde{b}\tilde{y})$, we obtain a Bell inequality for $N$ parties

$$\sum_{a_1...a_k,x_1...x_k} c_{a_1...a_k,x_1...x_k} P(a_1...a_k\tilde{b}|x_1...x_k\tilde{y}) \leq kP(\tilde{b}\tilde{y}). \tag{6.5}$$

which will be violated by the original state $\rho_{A_1...A_N}$. In Fig. 6.2 we illustrate the content of this observation for the tripartite case.

### 6.2.1 Quantum nonlocality and one-way entanglement distillation.

Observation 1 can be used to obtain a direct link between one-way entanglement distillability and nonlocality.

Before that, let us start by considering the following situation. Consider the isotropic state $\rho^{(d)}_{A_1...A_N}$ (6.2) in a region of parameters where it is local and entangled. The goal is to reveal its nonlocality by distributing copies of it by several parties and using Observation 1. Take then the simplest multipartite case of three parties -- Alice, Bob, and Charlie --, and imagine that Bob shares one copy of this state with Alice, $\rho^{(d)}_{AB_1}$, and another copy with Charlie, $\rho^{(d)}_{CB_2}$. If Bob makes the dichotomic measurement defined by the operators

$$P_+ = |\phi^+_d\rangle \langle \phi^+_d| \tag{6.6}$$

$$P_- = 1 - |\phi^+_d\rangle \langle \phi^+_d| \tag{6.7}$$

1See details in section 2.3.4
6.2. NONLOCALITY REVEALED BY Multipartite Strategy

Figure 6.2: Illustration of Observation 1. Three parties, Alice, Bob and Charlie, share a quantum state $\rho_{ABC}$ on which they perform measurements $M_A$, $M_B$, and $M_C$ respectively. After Charlie applies measurement $M_C$ and gets the outcome $c$, Alice and Bob are left with a state $\rho$. If this state is nonlocal, i.e. if there exist measurements $M_A$ and $M_B$ such that $P(ab|M_A M_B, M_C = c)$ cannot be written as (2.21), then the original state $\rho_{ABC}$ is also nonlocal.

and obtains the outcome associated to the projection on the maximally entangled state, the systems $A$ and $C$ are left in same state $\rho_{AC}^{(d)}$. In this scenario, no gain comes from this: from Observation 1, we can only conclude that the original state $\rho_{AB_1}^{(d)} \otimes \rho_{CB_2}^{(d)}$ is nonlocal only if $\rho^{(d)}$ was already nonlocal.

But now, give the parties many copies of the state, $(\rho_{AB_1} \otimes \rho_{CB_2}) \otimes N$. A possible strategy is the following (see Fig. 6.3). Bob applies a measurement in the part of the state he shares with Alice such that, for a given outcome, they are left in a maximally entangled state. Then, he repeats the same procedure with the state shared with Charlie. When he is successful, he will share a maximally entangled state with Alice and another with Charlie. Now, by performing the measurement (6.6), Bob will sometimes project Alice and Charlie's systems in a maximally entangled state, which is obviously nonlocal. Although this strategy was presented in a sequential way, it can be seen as a single two-outcome measurement applied by Bob, such that for one of the outcomes Alice
and Charlie are left with a nonlocal state. This proves that the original state \((\rho_{AB} \otimes \rho_{CB})^\otimes N\) is nonlocal, and therefore the nonlocality of \(\rho^{(d)}\) has been activated.

The key step in previous construction is the possibility of transforming many copies of a given state into a maximally entangled state by a measurement in only one of the parties. This is indeed equivalent to the problem of one-way entanglement distillation [HHHH09]. We can relax the assumptions above and consider more general quantum states \(\rho_{AB}\) and \(\rho'_{CB}\). Consequently, we obtain a direct connection between one-way entanglement distillability and nonlocality in the multipartite scenario:

**Observation 2.** If \(\rho\) and \(\rho'\) are one-way distillable entangled states, the state \((\rho_{AB} \otimes \rho'_{BC})^\otimes N\) is nonlocal for \(N\) large enough.

Let us stress the role of using multipartite systems in the present nonlocality test. First, entanglement distillation is not equivalent to nonlocality for a given state in the bipartite scenario. Indeed, the entanglement of isotropic states (6.2) is distillable for some range of the noise parameter \(p\) where it is local (see Fig. 6.1). The key point is that the subsystems where the preparation measurements are performed need to be separated from the subsystems used to test nonlocality. Otherwise, it is not possible to guarantee the independence condition (6.4). Therefore, observing nonlocality of the post measurement state, even if it occurs for all outcomes of the initial measurements, is not enough to infer the nonlocality of the original state. However, in our multipartite scenario, one-way entanglement distillation can indeed be used to detect nonlocality. It follows from the fact that the preparation measurements and Bell measurements define spacelike separated events.

Returning to the case where the parties share isotropic states (6.2), we use Observation 2 to improve the noise threshold \(p\) for which the states are known to be nonlocal. For that, we consider a sufficient criterion for a given bipartite state \(\rho_{AB}\) to be one-way distillable, introduced in [DW05]. It says that \(\rho_{AB}\) is one-way distillable if \(S(\rho_B) - S(\rho_{AB}) > 0\), where \(S\) is the von Neumann entropy of the quantum state and \(\rho_B\) its reduced density matrix. It follows that the state (6.2) is one-way distillable for \(p > 1/2\) in the limit of large \(d\). This is well below the nonlocality threshold known so far, which was \(p = 0.67\) and corresponded to the violation of the CGMLP inequality (see Fig. 6.1).

### 6.2.2 Activation of bipartite nonlocality

In the case of two-qubit isotropic states, \(\rho^{(2)}\), the previous strategy detects nonlocality for \(p \gtrsim 0.74\), and therefore does not improve the nonlocality threshold given by the violation of the CHSH inequality. However, following a procedure from Ref.[SSB⁺05], it is possible to significantly improve this bound and show that \(N\) copies of \(\rho^{(2)}\) violate a Bell inequality for \(p \gtrsim 0.64\). Notice that this was basically obtained in [SSB⁺05], but there the authors did not take into account Observation 1. and argued that the type of nonlocality activated was not equivalent to standard one.

The example of [SSB⁺05] considers several copies of the isotropic state \(\rho^{(2)}\) distributed in a star configuration, as depicted in Fig. 6.4. A central party, \(A_0\), shares a copy of \(\rho^{(2)}\) with each one of the \(N\) separated parties \(A_1, \ldots, A_N\).\(^2\)

\(^2\)See discussion in section 2.3.2.
Consider three space-like separated parties Alice, Bob, and Charlie. Alice and Bob share many (ideally infinite) copies of an entangled state $\rho_{AB}$. In the same way, Bob and Charlie share many copies of some other state $\rho'_{BC}$. A possible strategy to detect the nonlocality of their system is the following. First Bob applies a measurement in the subsystems he shares with Alice, as well as in those ones he shares with Charlie (upper panel). If the states they share are one-way entanglement distillable, there is a probability that after the measurements, Alice-Bob and Bob-Charlie, end up sharing maximally entangled states (middle panel). In this case, Bob finishes the protocol by performing the dichotomic measurement (6.6) which, for one of the outcomes, projects Alice and Charlie’s state into a maximally entangled state (lower panel). This protocol can also be seen as if Bob had applied a single dichotomic measurement that projects the state of Alice and Charlie into a maximally entangled state with a given probability. Since this state is clearly nonlocal, we conclude that the tripartite starting state must also be nonlocal (Observation 1). If $\rho_{AB}$ and $\rho'_{BC}$ are set to be isotropic states (6.2), this strategy allows the detection of nonlocality for $p > 1/2$ and large $d$. In the case of a large, but still finite, number of copies, the state $\rho_{AB}^\otimes N$ (or $\rho_{BC}^\otimes N$) is not distilled into a perfect maximally entangled state, but into an approximation of it. The protocol also works in this case, provided that the final state shared by Alice and Charlie is nonlocal.

If $A_0$ projects its system in the GHZ-state, $|GHZ\rangle = (|0\ldots0\rangle + |1\ldots1\rangle)/\sqrt{2}$, the remaining parties will be left with a state that violates a functional Bell inequality for $p > (2/\pi)2^{1/N}$ [SSB+05]. From Observation 1, this implies that the original quantum state, formed by the tensor product of copies of $\rho^{(2)}$, is nonlocal for $p > (2/\pi)2^{1/N}$. Note that this threshold goes to $2/\pi \approx 0.64$. 

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Figure 6.3: **One-way entanglement distillation and multipartite nonlocality.** Consider three space-like separated parties Alice, Bob, and Charlie. Alice and Bob share many (ideally infinite) copies of an entangled state $\rho_{AB}$. In the same way, Bob and Charlie share many copies of some other state $\rho'_{BC}$. A possible strategy to detect the nonlocality of their system is the following. First Bob applies a measurement in the subsystems he shares with Alice, as well as in those ones he shares with Charlie (upper panel). If the states they share are one-way entanglement distillable, there is a probability that after the measurements, Alice-Bob and Bob-Charlie, end up sharing maximally entangled states (middle panel). In this case, Bob finishes the protocol by performing the dichotomic measurement (6.6) which, for one of the outcomes, projects Alice and Charlie’s state into a maximally entangled state (lower panel). This protocol can also be seen as if Bob had applied a single dichotomic measurement that projects the state of Alice and Charlie into a maximally entangled state with a given probability. Since this state is clearly nonlocal, we conclude that the tripartite starting state must also be nonlocal (Observation 1). If $\rho_{AB}$ and $\rho'_{BC}$ are set to be isotropic states (6.2), this strategy allows the detection of nonlocality for $p > 1/2$ and large $d$. In the case of a large, but still finite, number of copies, the state $\rho_{AB}^\otimes N$ (or $\rho_{BC}^\otimes N$) is not distilled into a perfect maximally entangled state, but into an approximation of it. The protocol also works in this case, provided that the final state shared by Alice and Charlie is nonlocal.

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---
for large $N$, that surprisingly enters the region of values of $p$ for which a local model for projective measurements is known to exist [AGT06] (see Fig.6.1). Now, two scenarios would be possible: (i) the states $\rho^{(2)}$ are nonlocal for generalized measurements for $p \geq 2/\pi \sim 0.64$ or, (ii) $\rho^{(2)}$ are local under single general measurements for some $0.64 \leq p \leq 0.66$ and our example proves that its nonlocality can be activated through collective measurements.

However, the initial state $\rho_{A_0 \ldots A_N}$ violates a Bell inequality for $N + 1$-partite states, where the parties only need to perform dichotomic measurements. This follows from (6.3) and the fact that the Bell inequality violated by the projected state of parties $A_1 \ldots A_N$ is a correlation inequality. Moreover, any local model for projective measurements can be extended to two-outcome measurements (see below). Consequently, we can argue that we are in fact observing activation of multipartite nonlocality, as performing collective dichotomic measurements reveals the nonlocality of the isotropic states, in a region where it is local under these kind of measurements.

Any two-outcome general measurement can be simulated by projective measurements

Here we show that projective measurements are enough to simulate any correlations obtained by two-outcome general measurements. This indeed extends the local model reproducing the outcomes of projective measurements applied to $\rho^{(2)}$ to the case of two-outcome general measurements.

Consider a given measurement with elements $M_0$ and $M_1$. Take the spectral decomposition of $M_0$ as $M_0 = \sum_i \lambda_i |\varphi_i\rangle\langle\varphi_i|$. Given that $M_1 = 1 - M_0$, we can diagonalize $M_1$ in the same basis as $M_0$. Consequently, the results of this general measurement can be simulated by the projective measurement defined by the projectors $|\varphi_i\rangle\langle\varphi_i|$.

6.2.3 Activation of genuine multipartite nonlocality

Finally, we give examples which show that genuine multipartite nonlocality can be activated. For that, we use the sufficient criterion for the presence of fully multipartite nonlocality enunciated in chapter 5, along with Observation 2. All our examples consider the completely no-signaling scenario, as commented before. First, we show that copies of $m$-partite states ($m < N$), distributed among $N$ parties, can exhibit $N$-partite full nonlocality. Then, we provide an example of a tripartite quantum state $\rho$, which contains only bipartite nonlocality, such that $\rho^{\otimes N}$ is genuinely tripartite nonlocal.

Two singlets with tripartite full nonlocality

The first example turns out to be surprisingly simple. It considers two copies of a maximally-entangled state, $|\phi^+\rangle = 1/\sqrt{d} \sum_{i=1}^{d} |ii\rangle$, shared by three parties $A$, $B$ and $C$

$$|\psi\rangle_{ABC} = |\phi^+\rangle_{AB} |\phi^+\rangle_{AC}.$$  \hspace{1cm} (6.8)

Given that there is a maximally entangled state across any bipartition, the sufficient condition for presence of full-multipartite nonlocality, from chapter 5,
6.2. NONLOCALITY REVEALED BY Multipartite STRATEGY

Figure 6.4: Activation of bipartite nonlocality. Consider the scenario where \( N + 1 \) parties share quantum states \( \rho^{(2)} \) in a star-like configuration. The central party applies a dichotomic measurement described by the operators \( \{ |GHZ\rangle\langle GHZ|, \mathbb{1} - |GHZ\rangle\langle GHZ| \} \), where \( |GHZ\rangle = (|0\ldots0\rangle + |1\ldots1\rangle)/\sqrt{2} \) is the well-known GHZ state. In the case where central subsystem is projected into the GHZ state, the remaining parties are left in a state that is nonlocal for \( p > (2/\pi)2^{1/N} \) [SSB'05]. This bound goes to \( p \sim 0.64 \) as \( N \to \infty \). There exists a local model describing the results of any von Neumann projective measurement applied to \( \rho^{(2)} \) for \( p \leq 0.66 \) [AGTO06]. Moreover, this model can be extended for the case of two-outcome general measurements, which proves the activation of nonlocality of \( \rho^{(2)} \) in this example.
applies and the state has $p_{NS} = 1$. Therefore, it violates a Svetlichny-like Bell inequality for the detection of genuine multipartite nonlocality. The example naturally generalizes to any number of parties: the state $|\phi^+\rangle_{AB_1} |\phi^+\rangle_{AB_2} \ldots |\phi^+\rangle_{AB_{N-1}}$ is genuine $N$-partite nonlocal.

Side remark. The example of fully multipartite nonlocal mixed state presented in chapter 5 is also an example of activation of genuine multipartite nonlocality. There, we have seen that five copies of the four-partite Smolin state form a fully five-partite nonlocal state.

6.2.4 Activation of genuine nonlocality

Finally, we provide a strongest example of activation of genuine multipartite nonlocality, in which $N$ copies of a state $\rho$ become multipartite nonlocal. It is built upon our first example and considers the tripartite mixed state

$$\rho = \frac{1}{2} (\phi_{AB}^+ \otimes \mathbf{1}_C \otimes |000\rangle \langle 000|_{A'B'C'} + \phi_{AC}^+ \otimes \mathbf{1}_B \otimes |111\rangle \langle 111|_{A'B'C'}), \quad (6.9)$$

with $\phi^+ = |\phi^+\rangle |\phi^+\rangle$. Clearly, it follows from convexity that the state $\rho$ contains only bipartite nonlocal correlations. We show in what follows that $\rho^{\otimes N}$, for sufficiently large $N$, is genuine tripartite nonlocal. In order to do that, it is convenient to interpret the qubits in systems $A'$, $B'$ and $C'$ as flags. Recall that the state (6.8) has tripartite nonlocality: there exist local measurements by the parties, $M_a^a$, $M_b^b$ and $M_c^c$, acting on this state such that the corresponding correlations,

$$P_\psi(abc | xyz) = \text{tr}(|\psi\rangle \langle ABC| M_a^a \otimes M_b^b \otimes M_c^c), \quad (6.10)$$

violate a Svetlichny-like inequality.

Based on that, consider the correlations resulting from applying the following measurements on $N$ copies of the state (6.9): Alice measures the $N$ flags $A'$. She keeps the two systems in $A$ where the measurement result on the flags was 0 and 1 for the first time. She knows that these systems correspond to singlets shared with Bob and Charlie, depending on the flag result. Thus, she applies on these systems the measurement $M_a^a$ of (6.10). Bob and Charlie apply a similar strategy: Bob (Charlie) measures the flags and applies the measurement $M_b^b$ ($M_c^c$) of Eq. (6.10) on the system $B$ ($C$) corresponding to the first instance in which the flag $B'$ ($C'$) was measured to be in 0 (1). Although presented in a sequential way for clarity reasons, this process defines in fact one-shot local measurements on each party. Clearly, the obtained correlations are the same as in (6.10). There exists however a probability $p_{eq} = 1/2^{N-1}$ that all the flags provide the same result. Assume for instance that all the flags give 0. Then, the parties follow, for instance, the measurement strategy on $N$ singlets $|\phi^+\rangle_{AB}$ which optimally approximates $P_\psi(abc | xyz)$. The obtained bipartite nonlocal correlations are denoted by $\tilde{P}_\psi(abc | xyz)$ although its explicit form is irrelevant for our considerations. Putting all the possibilities together, the total correlations among the parties are equal to

$$p_{eq} \tilde{P}_\psi(abc | xyz) + (1 - p_{eq}) P_\psi(abc | xyz) \quad (6.11)$$

which can be made arbitrarily close to $P_\psi(abc | xyz)$ as $p_{eq}$ tends to zero exponentially with the number of copies $N$. Therefore, there always exists a finite value
of $N$ (that might not be large), such that genuine tripartite nonlocal correlations can be obtained from $\rho^{\otimes N}$, although $\rho$ did not contain this resource.

### 6.3 Discussion

In the present study we have seen how considering Bell tests in a multipartite scenario can be a powerful tool to study both bipartite and multipartite nonlocality. Namely, we use the fact that in order to prove the nonlocality of a multipartite state, it is sufficient to show that there is a measurement in a subset of its parties that, for some outcome, projects the remaining parties on a nonlocal quantum state. We apply this criterion on multipartite quantum states composed of several copies of bipartite quantum states, distributed by the parties according to specific configurations. Following this technique we obtain a variety of results. First, we could provide a direct link between nonlocality and one-way entanglement distillation, two a priori unrelated concepts. With this, we considerably lower the threshold for violation of a Bell inequality for isotropic states, in the asymptotic limit of $d \to \infty$.

Second, we are able to show that nonlocality can be activated by performing collective measurements on copies of some quantum state. In the bipartite case, we use an example provided by [SSB+05] to show that, taking $N \to \infty$ copies of the isotropic state for $d = 2$, it is possible to reveal the nonlocality of the state in a region for which it is known to be local under projective and dichotomic general measurements. In the multipartite case, we construct examples of quantum states without any genuine multipartite nonlocality but which, when combined, are genuinely multipartite nonlocal. In particular, we prove that two copies of a singlet state distributed by three parties is sufficient to obtain fully tripartite nonlocal correlations, which is activation in its strongest form (from $p_{\text{NS}} = 0$ to $p_{\text{NS}} = 1$). And finally, we provide an example of a tripartite quantum state $\rho$, with bipartite nonlocality, such that $\rho^N$ is genuine tripartite nonlocal, for some $N$ not necessarily very large.

The main open questions arising from this work are the following. Can the link between one-way entanglement distillation and nonlocality be applied to show the nonlocality of a broader class of quantum states, which do not violate any of the available known Bell inequalities? Is it possible to improve the results concerning the activation of bipartite nonlocality, in a way that the nonlocality threshold enters the region of existence of local model for any single copy measurements? This would provide a definite proof of the activation of bipartite nonlocality. Finally, would our results on activation of genuine nonlocality hold for the original framework of Svetlichny, which considered that some patterns of communication between the parties?
Chapter 7

Guess your neighbour’s input: a multipartite nonlocal game with no quantum advantage

We present a multipartite nonlocal game in which each player must guess the input received by his neighbour. We show that quantum correlations do not perform better than classical ones at this game, for any prior distribution of the inputs. There exist, however, input distributions for which general no-signaling correlations can outperform classical and quantum correlations. Some of the Bell inequalities associated to our construction correspond to facets of the local polytope. These results suggest that quantum correlations might obey a generalization of the usual no-signaling conditions in a multipartite setting.

7.1 Introduction

In recent years, the study and understanding of quantum nonlocality – the fact that certain quantum correlations violate Bell inequalities [Bel64] – has benefited from a cross-fertilization with information concepts. If in one hand, nonlocality has been identified as a key resource for quantum information processing, in the other, information concepts provided a deeper understanding of the nature of quantum nonlocality.

Up to now, such results are arguably incomplete, in the sense that they almost exclusively refer to the bipartite scenario. Here our aim is to investigate the separation between quantum and no-signaling correlations in a multipartite scenario. For this, we introduce and study a simple multipartite nonlocal game, Guess Your Neighbour’s Input (GYNI).

In GYNI, $N$ distant players are arranged on a ring and each receive an input bit $x_i \in \{0, 1\}$ (see Fig. 1). The goal is that each participant provides an output bit $a_i \in \{0, 1\}$ equal to its right neighbour’s input bit:

$$a_i = x_{i+1} \quad \text{for all } i = 1, \ldots, N, \quad (7.1)$$

See Motivation in the Introduction.
Figure 7.1: Representation of the GYNI nonlocal game. The goal is that each party outputs its right-neighbour’s input: $a_i = x_{i+1}$.

where $x_{N+1} \equiv x_1$. The $2^N$ possible input strings $x = (x_1, \ldots, x_N)$ are chosen according to some prior distribution $q(x) = q(x_1, \ldots, x_N)$, which is known to the parties. The figure of merit of the game is given by the average winning probability

$$\omega = \sum_x q(x) P(a_i = x_{i+1} | x),$$

where $P(a_i = x_{i+1} | x) = P(a_1 = x_2, \ldots, a_N = x_1 | x_1, \ldots, x_N)$ denotes the probability of obtaining the correct outputs (7.1) when the players have received the input string $x$. Of course, players are not allowed to communicate after the inputs are distributed. Thus, their performance depends only on the initially agreed strategy and on the shared physical resources.

The GYNI game captures a particular notion of signaling: if the players were able to win with high probability, their output would reveal some information about their neighbour’s input. We therefore expect that the nonlocal correlations of quantum theory cannot be exploited by non-communicating observers to perform better at GYNI than using classical resources alone. We confirm this intuition and prove that, indeed, quantum correlations provide no advantage over classical correlations. Surprisingly, however, the no-signaling principle is not at the origin of the quantum limitation: for $N \geq 3$, there exist input distributions $q$ for which no-signaling correlations provide an advantage over the best classical and quantum strategies. This suggests the possibility that in a multipartite scenario, quantum correlations obey a qualitatively stronger version of the usual no-signaling conditions.

Each of the input distributions $q$ associated with a non-trivial no-signaling strategy defines a Bell inequality whose maximal classical and quantum values coincide, but whose no-signaling value is strictly larger. Interestingly, some of these inequalities define facets of the polytope of local correlations. We thus prove the existence of non-trivial facet Bell inequalities with no quantum violation, answering a question raised by Gill [Gil05]. Moreover, since these Bell inequalities are facets, the GYNI game identifies a portion of the boundary of the set of quantum correlations of non-zero measure, in contrast with previous information-theoretic or physical limitations on nonlocality [vD05, BBL+06, LPSW07, PlöPK+09, BS09, SBP09, ABPS09, NW10].
7.2 GYNI with classical and quantum resources

We start by showing that the optimal classical and quantum winning strategies are identical for any prior distribution $q$ of the inputs. Let us first show that there is a simple classical strategy achieving a winning probability

$$\omega_c = \max_x [q(x) + q(\overline{x})], \quad (7.3)$$

where $\overline{x}$ denotes the “negation” of the input string $x$, $x = (\bar{x}_1, \ldots, \bar{x}_N)$ with $\bar{x}_i = x_i \oplus 1$, and $\oplus$ denotes addition modulo 2. This strategy is based on the following simple observation.

Let $y$ be an arbitrary string. If $x \neq y, \overline{y}$, there exists an $i$ s.t. $x_i = y_i$ and $x_i+1 \neq y_i+1$.

Indeed, if this was not the case, we would have that for any $i$, either $x_i \neq y_i$ or $x_i+1 = y_i+1$. But this would in turn imply that either $x = y$ or $x = \overline{y}$, in contradiction with the hypothesis.

Consider now a classical strategy specified by the string $y$, where each party outputs the bit $a_i = y_{i+1}$ if it received the input $y_i$, and outputs $a_i = y_i+1$ if it received $\overline{y}_i$. It obviously follows that $P(a_i = y_{i+1}|y) = 1$ and $P(a_i = y_{i+1}|\overline{y}) = 1$. On the other hand, $P(a_i = x_{i+1}|x) = 0$ for all $x \neq y, \overline{y}$. Indeed, from observation (7.4), there exists an $i$ such that $x_i = y_i$, but for which $a_i = y_{i+1} \neq x_{i+1}$. The winning probability of this classical strategy is thus equal to $\omega = q(y) + q(\overline{y})$, which yields (7.3) if we take $y$ to be $q(y) + q(\overline{y}) = \max_x [q(x) + q(\overline{x})]$.

We now prove that there is no better quantum (hence classical) strategy. In the most general quantum protocol, the parties share an entangled state $|\psi\rangle$ and perform projective measurements on their subsystem dependent on their inputs $x_i$. They then output their measurement results $a_i$. Denoting $M_{a_i}$ the projection operator associated to the output $a_i$ for the input $x_i$, the probability that the $N$ players produce the correct output is thus given by

$$P(a_1 = x_2, \ldots, a_N = x_N|x_1, \ldots, x_N) = \langle M_{x_2}^{a_1} \otimes \ldots \otimes M_{x_N}^{a_N} \rangle,$$

and the average winning probability is

$$\omega = \sum_x q(x) \langle M_x \rangle, \quad (7.5)$$

where we have written $M_x = M_{x_2}^{a_1} \otimes \ldots \otimes M_{x_N}^{a_N}$ for short. The operators $M_x$ satisfy the following properties

$$M_x^2 = M_x, \quad (7.6)$$

and

$$M_x M_y = 0 \quad \text{if } x \neq y, \overline{y}. \quad (7.7)$$

The first property follows from the fact that the $M_x$ are projection operators. The second property follows from the orthogonality relations $M_x^{a_i} M_{x_i}^{a_i} = 0$ and observation (4). Note that protocols involving mixed states or general measurements can all be represented in the above form by expanding the dimensionality of the initial state.

We now show, using (7.6) and (7.7), that $\omega = \sum_x q(x) M_x \leq \omega_c$, where $\leq$ should be understood as an operator inequality, i.e., $A \leq B$ means that
First note that \( \sum_q q(x) M_x \leq \sum_q q'(x) M_x \), where \( q'(x) = q(x) + (\omega_c - q(x) - q(x))/2 \) since by definition \( \omega_c - q(x) - q(x) \geq 0 \). It is thus sufficient to consider weights \( q \) such that \( q(x) + q(x) = \omega_c \) for all \( x \). We can then write

\[
\omega_c - \sum_x q(x) M_x = \left[ \sqrt{\omega_c} - \sum_x \alpha_x M_x \right]^2 + \frac{1}{2} \sum_x [\beta_x M_x - \beta_x M_x]^2 \tag{7.8}
\]

where \( \alpha_x = \sqrt{\omega_c} - q(x)/\sqrt{\omega_c} \) and \( \beta_x = \sqrt{q(x)q(x)/\omega_c} \). To verify this identity we only need to use \( (7.6), (7.7) \), and the fact that \( q(x) + q(x) = \omega_c \). Note now that the right hand-side of \( (7.8) \) is \( \geq 0 \), since it is a sum of square involving only hermitian operators. This shows that \( \sum_q q(x) M_x \leq \omega_c \), as announced.

The inequality \( \sum_q q(x) P(a_i = x_{i+1} | x) \leq \omega_c \) can be interpreted as a Bell inequality whose local and quantum bound coincide. It is well known that in order to achieve a Bell violation in quantum theory one must perform measurements corresponding to non-commuting operators. The above proof, however, does not distinguish non-commuting operators from ordinary, commuting numbers: it is based on the algebraic identity \( (7.8) \) which follows only from Eqs. \( (7.6) \) and \( (7.7) \), regardless of whether the \( M_x \)'s commute or not. This explains why the classical and quantum bounds are identical.

### 7.3 GYNI with no-signaling resources

At first sight, it may seem that the quantum limitation on the GYNI game arises from the no-signaling principle: if the players were able to win with high probability, their output would somehow depend on their neighbour’s input. This motivates us to look at how players constrained only by the no-signaling principle perform at GYNI.

Formally, the no-signaling principle states that the marginal distribution \( P(a_1, \ldots, a_{i_k} | x_{i_1}, \ldots, x_{i_k}) \) for any subset \( \{i_1, \ldots, i_k\} \) of the \( n \) parties should be independent of the measurement settings of the remaining parties [BLM+05], i.e., that

\[
P(a_1, \ldots, a_{i_k} | x_{i_1}, \ldots, x_{i_k}) = P(a_{i_1}, \ldots, a_{i_k} | x_{i_1}, \ldots, x_{i_k})
\]

This guarantees that any subset of the parties is unable to signal to the other by their choice of inputs.

We show in App. C.1 that players constrained only by no-signaling have a bounded winning probability \( \omega_{ns} \leq 2\omega_c \). They thus cannot win in general with unit probability at GYNI. Furthermore, for certain input distributions, such as the one where all input strings are chosen with equal weight \( q(x) = 1/2^N \), we show as expected that \( \omega_{ns} = \omega_c \). That is, for uniform and completely uncorrelated inputs, any resource performing better than a classical strategy is necessarily signaling.

Surprisingly, this property is not general. There exist distributions \( q(x) \) for which no-signaling strategies outperform classical and quantum strategies.
Consider for instance the following input distribution

\[
q(x) = \begin{cases} 
1/2^{N-1} & \text{if } x_1 \oplus \cdots \oplus x_\hat{N} = 0 \\ 
0 & \text{otherwise},
\end{cases} 
\]  

(7.9)

where \( \hat{N} = N \) if \( N \) is odd and \( \hat{N} = N - 1 \) if \( N \) is even. It easily follows from the previous analysis that for classical and quantum resources, \( \omega_c = 1/2^{N-1} \).

We now prove, however, that no-signaling resources can achieve \( \omega_{ns} = 4/3 \omega_c \). Note that the distribution (7.9) can be interpreted as a promise that the sum of the inputs (modulo 2) is equal to zero. This prior knowledge does not yield any information to the parties about the value of their neighbour's input, yet it can be exploited by no-signaling correlations to outperform classical strategies.

We start by considering the case \( N = 3 \), for which

\[
\omega = \frac{1}{4} [P(000|000) + P(110|011) + P(011|101) + P(101|110)],
\]  

(7.10)

where \( P(000|000) = P(a_1 = 0, a_2 = 0, a_3 = 0|x_1 = 0, x_2 = 0, x_3 = 0) \), and so on. Consider the first three terms in (7.10). The no-signaling principle implies that

\[
P(000|000) \leq \sum_{a_3} P(00a_3|000) = \sum_{a_3} p(00a_3|001),
\]

(7.11a)

\[
P(110|011) \leq \sum_{a_2} P(1a_2|011) = \sum_{a_2} p(1a_2|001),
\]

(7.11b)

\[
P(011|101) \leq \sum_{a_1} P(a_11|101) = \sum_{a_1} p(a_11|001).
\]

(7.11c)

By normalization of probabilities, the sum of the right-hand sides of Eqs. (7.11) is upper-bounded by one, and thus \( P(000|000) + P(110|011) + P(011|101) \leq 1 \). Similar conditions are obtained for any of the four possible combination of three terms in Eq. (7.10). Summing over these possibilities, we find \( 3[P(000|000) + P(110|011) + P(011|101) + P(101|110)] \leq 4 \), or in other words \( \omega_{ns} \leq 4/3 \times 1/4 = 4/3 \omega_c \). Furthermore the inequality is saturated only if the four probabilities appearing in (7.10) are all equal to 1/3. It turns out that the remaining entries of the probability table \( P(a|x) = P(a_1a_2a_3|x_1x_2x_3) \) can be completed in a way that is compatible with the no-signaling principle, i.e, the bound \( \omega_{ns} \leq 4/3 \omega_c \) is achievable. Up to relabeling of inputs and outputs, there exist two inequivalent classes of extremal no-signaling correlations achieving this winning probability (see App. C.2). One of them takes the form \( P(a|x) = 2/3 g(a,x) + 1/3 g'(a,x) \) where \( g \) and \( g' \) are the following boolean functions

\[
g(a,x) = a_1a_2a_3(1 \oplus x_1)(1 \oplus x_2)(1 \oplus x_3)
\]

\[
g'(a,x) = (1 \oplus a_1)(1 \oplus a_2)(1 \oplus a_3)
\]

\[
\oplus x_1a_2a_3 \oplus a_1x_2a_3 \oplus a_1a_2x_3 \oplus x_1x_2x_3.
\]

(7.12)

From this definition, it is easy to verify that \( P(a_1a_2a_3|x_1x_2x_3) \) satisfies the no-signaling conditions and achieves winning probability \( \omega_{ns} = 1/3 = 4/3 \omega_c \).

The existence of no-signaling correlations achieving \( \omega_{ns} = 4/3 \omega_c \) in the case \( N = 3 \) is enough to show that \( \omega_{ns} \geq 4/3 \omega_c \) for any \( N \geq 3 \). This can be seen as
follows. Consider the situation in which the first three parties use the optimal strategy for \( N = 3 \) while the remaining parties simply output their input. In this case, all the terms in \( \omega \) vanish, except the four terms \( P(000,0\ldots0|000,0\ldots0) \), \( P(110,0\ldots0|011,0\ldots0) \), \( P(011,1\ldots1|101,1\ldots1) \), and \( P(101,1\ldots1|110,1\ldots1) \), which are all equal to \( 1/3 \).

Beyond these analytical results, we obtained the maximal no-signaling values of \( \omega_{ns} \) up to \( N = 7 \) players using linear programming. The ratios \( \omega_{ns}/\omega_c \) of no-signaling to classical winning probabilities are \( 4/3 \) for \( N = 3 \), \( 4/3 \), \( 16/11 \) for \( N = 5 \), \( 6/5 \), and \( 64/42 \) for \( N = 7 \), showing that for more parties there exist no-signaling correlations that can outperform the optimal no-signaling strategy for \( N = 3 \). (Note that it can be shown that the winning probability for an odd number \( N \) of parties is always equal to the winning probability for \( N+1 \) players, see App. C.3).

### 7.4 GYNI Bell inequalities

The GYNI Bell inequalities \( \sum_x q(x)P(a_i = x_{i+1}|x_i) \leq \omega_c \) are not violated by quantum theory, but can be violated by more general no-signaling theories. In [Gil05], Gill raised the question of whether there exist Bell inequalities which (i) feature this 'no quantum advantage' property and (ii) define non-trivial facets of the polytope of local correlations. Here we give a positive answer to this question. We have checked that the GYNI inequalities defined by the distribution (7.9) are facet-defining for \( N \leq 7 \) players. More generally, we verified that the inequalities defined by the distribution \( q(x) \) having uniform support on \( \bigoplus_{i=1}^N x_i = 0 \) are facet-defining for all \( N \leq 7 \). We conjecture that they are facet-defining for any number of parties. Note also that the polytope of local correlations for the case \( N = 3 \) (with binary inputs and outputs) was completely characterized in [Sli03], the inequality corresponding to (7.10) belongs to the class 10 of [Sli03]. Geometrically, our result shows that the polytope of local correlations and the set of quantum correlations have in common faces of maximal dimension (we recall that a facet corresponds to a \((d-1)\)-dimensional face of a \(d\)-dimensional polytope).

This also implies that GYNI is an information-theoretic game that identifies a portion of the boundary of quantum correlations which is of non-zero measure. To the best of our knowledge, all previously introduced information-theoretic or physical principles recovering part of the quantum boundary— including nonlocal computation [LPSW07], nonlocality swapping [SBP09], information causality [PloPK+09, ABPS09], and macroscopic locality [NW10]— were only able to single out a portion of zero-measure.

### 7.5 Discussion and open questions

In this chapter, we presented a multipartite nonlocal game for which local and quantum players perform equally well, but where no-signaling players can have a higher success probability. These results are based on the study of the Bell inequalities associated to the game, namely their local, quantum and no-signaling bounds. It happens that these multipartite inequalities have symmetry properties that make them quite peculiar: in general, computing any of these bounds
becomes very hard with the size of the space of no-signaling probability distributions. However, in our case, the particular orthogonality properties (7.7) arising from the choice of probability terms defining the GYNI inequalities is enough to conclude that, regardless the number of parties, the local and quantum bounds are always the same for any distribution of inputs. Moreover, we find a particular input distribution for which we guarantee that the no-signaling bound is strictly larger.

Our work raises plenty of new questions. First, it would be interesting to understand the structure of those input distributions \( q \) leading to a gap between no-signaling and classical/quantum correlations (See App. C.1, for a class of distributions for which there is no gap). For instance, in the case of four parties, the distribution \( q \) having uniform support on \( x_1 \oplus x_2 \oplus x_3 \oplus x_1 x_2 x_3 = 0 \) leads to \( \omega_{ns} = 4/3 \omega_c \). However, the corresponding Bell inequality is not a facet. Another question is thus to single out, among all relevant input distributions, those corresponding to facet Bell inequalities. For three parties, it follows from [Sli03] that the distribution (7.9) is the unique possibility.

A further interesting problem is whether there exist facet Bell inequalities with no quantum advantage in the bipartite case. Note that our GYNI inequalities are non-trivial only for \( N \geq 3 \); for the case \( N = 2 \), the classical and no-signaling bounds are always equal. In Ref. [LPSW07], examples of bipartite Bell inequalities with no quantum advantage have been presented in the context of nonlocal computation. However, as mentioned earlier, none of the Bell inequalities associated to nonlocal computation has been proven to be facet-defining. We studied this question here and could prove that none of the simplest inequalities from [LPSW07] (corresponding to the family of inequalities specified by the parameters \( n = 2, 3 \) in [LPSW07]) are facet inequalities. The proof uses a mapping from these inequalities to the space of correlation inequalities for \( n \) parties, two settings and two outcomes, which was fully characterized in Ref. [WW01a]; see App. C.4 for a detailed proof. We conjecture that none of the Bell inequalities introduced in [LPSW07] are facet-defining.

Coming back to our original motivation, it would be interesting to get a deeper understanding of the structure and information-theoretic properties of the no-signaling correlations giving an advantage over classical/quantum correlations, for instance those associated to inequality (7.10). In particular, it would be interesting to understand if they can be exploited for other information tasks. Finally, our results suggest that the quantum limitation on the GYNI game might originate from a generalization of the no-signaling principle in a multipartite setting. Can this intuition be made concrete? Are there more general information tasks with no quantum advantage?

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\(^2\) See App. A  
\(^3\) While completing this thesis, Winter (with contributions from Brunner, Skrzypczyk and myself) proved that, indeed, none of the NLC inequalities define facets of the local polytope.
CHAPTER 7. GUESS YOUR NEIGHBOUR’S INPUT: A MULTIPARITTE
NONLOCAL GAME WITH NO QUANTUM ADVANTAGE
Chapter 8

Overview and Future Perspectives

The results obtained during this thesis refer to the identification and characterization of resources for quantum processing tasks. This study also provided a deeper understanding of the structure of quantum theory.

We considered the problem of finding more efficient implementation methods to detect entangled quantum states. We studied physical approximations for positive maps, which are in the basis of the strongest criterion to detect entanglement, but which unfortunately are not directly implementable in the laboratory. For all considered cases, we observed that physical approximations to optimal positive maps lead to entanglement-breaking channels, which can be realized by measurement and state-preparation protocols. We conjecture that this holds in general, providing a particular simple experimental realization for the approximated positive maps. **Open questions.** A natural follow-up to this work is to prove/disprove such conjecture. It would also be interesting to construct specific entanglement detection schemes using the physically approximated positive maps.

Concerning the subject of quantum nonlocality, we address the question of its robustness in the presence of local noise. We prove that pure bipartite entangled quantum states, mixed with completely depolarized noise, loose their nonlocal properties before they become separable. Our results also show that local entangled states can be observed quite generally, and are not only present in ‘engineered’ families of states, as the Werner states. **Open questions.** Can we extend the study of robustness of quantum nonlocality to other kinds of local noise? In particular, to pure state local noise?

Using Bell tests in the multipartite scenario, we provide a new framework for the study of both bipartite and multipartite nonlocality. In one hand, we have seen that there exist quantum states that are multipartite fully-nonlocal, which means that they contain the strongest form of genuine nonlocality. These results were obtained in a totally no-signaling scenario, which differs from the original framework introduced for the study of genuine nonlocality. **Open questions.** Would our results be affected if some communication is allowed? Could those more general hybrid local-nonlocal models be able to describe some genuine quantum nonlocality, as defined in our framework?
In addition, using the multipartite framework, we could establish a link between one-way quantum distillation and nonlocality. With that, it was possible to increase the amount of noise for which we know that some noisy quantum states are able to violate a Bell inequality. *Open question.* Are there other quantum states whose nonlocal properties can be directly inferred using this reasoning?

Moreover, using the same kind of multipartite strategy, we finally prove that quantum nonlocal resources can be activated. This holds both for bipartite and genuine multipartite quantum correlations. We find examples of quantum states which do not possess these properties at a single copy level, but which manifest them when collective measurements on many copies are considered. *Open questions.* Is it possible to activate bipartite nonlocality in a bipartite scenario?

Finally, we extend the study of tasks that distinguish local/quantum correlations from post-quantum correlations to the multipartite scenario. We define a multipartite nonlocal game for which only stronger-than-quantum players are able to perform better than classical ones. This game is associated to facet Bell inequalities, therefore it identifies, for the first time, subsets of maximal dimension of the boundary of quantum correlations. Our nonlocal game is also naturally associated to signaling: high success probabilities indicate the presence of communication. The limitation of classical and quantum resources suggests then an underlying multipartite no-signaling principle. *Open questions.* Can the same kind of facet Bell inequalities be found in the bipartite case? Would these results be valid for generalizations of our nonlocal game? Can we formulate a generalized no-signaling principle that justifies the limitation of classical and quantum resources at this multipartite game?
Appendix A

The sets of local, quantum and no-signaling correlations

The study of quantum nonlocality is concerned with the characterization of correlations between the outcomes of measurements performed on spacelike separated quantum systems. If in one hand we are interested in deciding whether the correlations are genuinely quantum, that is, if they could not be obtained by any classical (local) theory, on the other hand, we would like to distinguish quantum correlations from other nonlocal no-signaling correlations. For both purposes, it is very useful to represent the outcome distributions in a probability space and study the geometrical properties of the sets formed by local, quantum and no-signaling correlations.

In this appendix, I introduce the basic concepts on the geometry of the sets of probability distributions, for which is enough to consider the bipartite scenario. So, as usual, we have distant parties Alice and Bob performing local measurements $x, y = 1, \ldots, m$ on their subsystems and obtaining outcomes $a, b = 1, \ldots, n$, distributed according to some probability distribution $P(ab|xy)$. This defines a vector in a $m^2 n^2$-dimensional space, with entries labeled by $abxy$.

The set of probability vectors $P(ab|xy)$ lives in a subspace defined by the normalization constraints

$$\sum_{a,b} P(ab|xy) = 1, \quad \forall x,y \tag{A.1}$$

and describes a convex polytope according to the positivity conditions

$$P(ab|xy) \geq 0, \quad \forall a,b,x,y \tag{A.2}$$

Usually, this probability polytope is called signaling polytope, to enhance the fact that general probability distributions $P(ab|xy)$ allow instant communication among the parties.

The no-signaling polytope. The no-signaling condition imposes further restrictions on the probability distributions $P(ab|xy)$. The fact that Alice cannot

\footnote{Again for simplicity reasons, I consider the particular case where Alice and Bob have access the same number $m$ of observables, with $n$ possible values. Obviously, the generalization for different number of inputs and outputs is straightforward.}
signal to Bob through her choice of local measurements is mathematically translated into

\[ P_{NS}(b|xy) = \sum_a P_{NS}(ab|xy) = P_{NS}(b|y), \forall b,x,y \]  

(A.3)

and similarly for Bob. These conditions define the no-signaling subspace, and the intersection of the signaling polytope with this subspace defines the no-signaling polytope.

**The quantum convex set.** According to the quantum theory, probability distributions are of the form

\[ P_Q(ab|xy) = \text{tr}(\rho M_a^x \otimes M_b^y), \]  

(A.4)

which means that outcome probabilities are given by local measurements described by operators \( M_a^x \) and \( M_b^y \) acting on the Hilbert space where some quantum state \( \rho \) is defined. Due to the linearity of the trace function and the quantum superposition principle, the points \( P_Q(ab|xy) \) form a convex set inside the no-signaling polytope, but which is not itself a polytope [Cir80].

**The local polytope.** Finally, the set of local (classical) correlations is described by probability distributions which admit a local model (2.21)

\[ P_L(ab|xy) = \int d\lambda \omega(\lambda) P^A(a|x,\lambda)P^B(b|y,\lambda). \]  

(A.5)

The local points \( P_L(ab|xy) \) describe a convex polytope inside the set quantum of correlations. It is completely defined by its extreme points, named deterministic points since they correspond to local deterministic outcomes, which means that the probabilities \( P_L(a|x) \) and \( P_L(b|y) \) can only assume the values 0 and 1 (see, for instance, [WW01b]).

A Bell inequality is a hypersurface on the space of probabilities, which divides the no-signaling subspace into halves: one that contains the local polytope and the other which therefore contains only nonlocal correlations (see figure A.3). It can be expressed by the condition

\[ \sum_{abxy} c_{abxy} P(ab|xy) \leq k, \]  

(A.6)

where \( c_{abxy} \) are real coefficients and \( k \) is a bound on the maximum value of the expression for any local theory. In the case where \( k \) is a tight bound, the inequality defines a face of the local polytope, i.e., a subset of its boundary. Moreover, when the face is of maximal dimension (it has dimension \( d-1 \), where \( d \) is the dimension of the no-signaling subspace), the Bell inequality describes a facet of the local polytope (see figure A.1). The local polytope can then either be represented by its vertices, or local deterministic points—the V-representation—, or by a finite set of facet Bell inequalities—the H-representation. Going from a representation to other consists of the convex hull problem, a standard problem in convex geometry. For the smallest dimensions, there exist efficient methods to obtain all the facet Bell inequalities, which means that it is possible to know exactly if a given distribution is local. In particular, for the case of two parties
A.1. NO-SIGNALING, QUANTUM AND LOCAL SETS IN THE PROBABILITY SPACE OF TWO PARTIES WITH TWO DICHTOMIC OBSERVABLES

Figure A.1: Representation of a region of the no-signaling space which contains a facet of the local polytope $L$, a portion of the convex set $Q$ of quantum correlations and a portion of the polytope of no-signaling distributions $NS$. The Bell inequality $I$ (represented by a full line) defines a face of the local polytope since it is tangent to this set in a subset of zero-measure. The local points of this intersection provide the local bound of the inequality. Analogously, the intersection between $I_Q$ ($I_{NS}$) with the quantum (no-signaling) region provides the maximum quantum (no-signaling) violation of $I$.

with two dichotomic observables, the local polytope is solely defined by CHSH-inequalities [CHSH69],[Fin82], and (trivial) positivity inequalities. A particular extension of this result for any number of parties was provided in [WW01a], where Werner and Wolf present a construction able to obtain all multipartite full correlation Bell inequalities, which are those that only include correlation terms with observables from all the parties involved in the experiment.

Given that a complete characterization of the local polytope soon becomes unfeasible for increased number of parties, observables and/or outcomes, it is of particular interest to know if a given Bell inequality defines a facet of the corresponding local polytope. For that, it is sufficient to check that the deterministic vectors that saturate the Bell inequality span the space of the boundary of the local polytope [Mas02].

A.1 No-signaling, quantum and local sets in the probability space of two parties with two dichotomic observables

Here I present the known results for the simplest non-trivial probability space. As mentioned before, in this case, the local polytope is completely described: a point of this space is nonlocal if and only if it violates some CHSH-inequality [CHSH69]. A complete description of the no-signaling polytope in terms of extreme no-signaling boxes was provided in [BLM"05], where it is shown that these vertices are generalizations of the PR-box (2.81). As for the quantum
boundary, no analytical expression that describes it has been discovered so far. In 1983, Cirel'son proved that the quantum violation of CHSH inequalities is never larger than $2\sqrt{2}$, known as the Cirel'son's bound\textsuperscript{2}, providing the first (very rough) approximation to the set of quantum correlations [Cir80]. Much later, in 2008, Navascués and co-authors derive a convergent method to obtain the set of quantum correlations, being able to obtain a much finer approximation of this set [NPA07]. Note that if one restricts to the no-signaling subspace with full correlations, an analytical expression for the quantum border was presented in [Mas03].

\textsuperscript{2}Interestingly, this bound is reached by Pauli measurements on a singlet state, showing that higher dimensional Hilbert spaces provide no advantage in this situation.
Appendix B

Unitary Symplectic Invariant States

Here we study the families of quantum states invariant under $SS$ and $S\bar{S}$ transformations, where $S$ is an unitary and symplectic operator defined on a Hilbert space of even dimension $d = 2n$. In general, we denote these by unitary symplectic invariant states, and represent them by $\rho_{SS}$ and $\rho_{S\bar{S}}$, respectively. To our knowledge, such operators have not been studied systematically as an independent family of states. They form a subfamily of $SU(2)$-invariant states of Ref. [Bre05], but since the number of parameters of the latter family increases with the dimensionality, it is manageable only for low dimensions.

In the following we obtain an explicit parametrization of unitary symplectic invariant states and consequently we find the set of parameters which define such states. Since both families are related by partial transposition, $\rho_{S\bar{S}} = \rho_{SS}^T$, we are able to find a region of parameters for which the states are PPT. Inside this set we identify the separable and the PPT-bound entangled regions, using symmetry considerations together with the Breuer-Hall map itself.

B.1 Parametrization of $\rho_{SS}$ and $\rho_{S\bar{S}}$

We start by considering the space of Hermitian $SS$-invariant operators, i.e. operators $A = A^\dagger$ such that

$$(S \otimes S)A(S^\dagger \otimes S^\dagger) = A.$$  \hspace{1cm} (B.1)$$

The invariance group $G$ is the intersection of the group $U(2n)$ of local unitaries, $U \otimes U$, with the group $Sp(2n)$ of local symplectic transformations, $Sp \otimes Sp$. The space of operators invariant under $G = Sp(2n) \cap U(2n)$ is then generated by the union of the set of basis operators from each invariance group alone [VW01] . As it is well known, $UU$-invariant operators correspond to Werner states and its space is spanned by the operators $I$ and $F$ [Wer89]. So, we need to find the form of $Sp(2n)$-invariant operators.

1 See Ref. [VW01] for a general theory of states invariant under the action of a group $G$.

2 In fact, the basis for the set of operators invariant under $G$ could be larger than the union of individual sets, but this is not the case since we are dealing with algebras (see [VW01]).
Let us then consider a symplectic operator $S$ (not unitary for the moment) and study its invariance under the action (B.1):

$$\sum_{j,l,m,n} S_{ij}S_{kl}A_{jlmn}S_{om}S_{pn} = A_{ikop}. \quad (B.2)$$

According to the definition of symplectic matrix (3.25), $S$ and its complex conjugation $\bar{S}$ are independent in general. But in order for (B.2) to hold, we need that the imaginary components to disappear. Therefore, the only possibility is for $A$ to be such that $ij = om \land kl = pn$ or $ij = pn \land kl = om$, which means that its non-zero entries are $A_{jil}$ or $A_{jil}$. Then, we have $A$ of the form $A_{jlmn} = \psi_{jlmn} = |\psi\rangle\langle\phi|$ and any linear combinations of similar terms. We can focus on rank-one matrices $A$ since higher ranks come directly by linearity. Also, since $A = A^\dagger$, we must have $A = |\psi\rangle\langle\psi|$. This leads to re-write Eq. (B.2) as

$$(S\psi S^T)_{ik}(S\psi S^T)_{po} = |\psi_{ik}\rangle|\psi_{po}\rangle. \quad (B.3)$$

But the only matrices that $S$ preserves are skew-symmetric, which implies that we must have $\psi_{ik} = cJ_{ik}$ for some complex scalar $c$ and symplectic matrix $J$. We choose $c = 1/\sqrt{d}$ and work in the Darboux basis such that $J$ is given by (3.26). Then we obtain

$$|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i,k} J_{ik}|ik\rangle = \frac{1}{\sqrt{d}} (|10\rangle - |01\rangle + |32\rangle - |23\rangle + \ldots) = (1 \otimes J)|\Phi_+\rangle. \quad (B.4)$$

Hence, $\Phi_d^J \equiv (1 \otimes J)\Phi_d^+(1 \otimes J^\dagger)$ is the only $Sp(2n)$-invariant operator, up to a multiplicative constant. Using this fact we conclude that the space of $SS$-invariant operators is spanned by $\{1, F, \Phi_d^J\}$. Correspondingly, performing partial transposition, we can easily obtain that the space of $SS$-invariant operators is spanned by $\{1, \Phi_d^+, F^J\}$, where $F^J \equiv (1 \otimes J)F(1 \otimes J^\dagger)$. To conclude, $SS$ and $SS$ invariant states can be respectively represented by

$$\rho_{SS} = \alpha_11 + \alpha_2F + \alpha_3\Phi_d^J \quad (B.5)$$

and

$$\rho_{SS} = \alpha_11 + \alpha_2F^J + \alpha_3\Phi_d^+ \quad (B.6)$$

where $\alpha_i$ are real parameters such that the states well-defined (positive) and normalized. Consequently, $\rho_{SS}$ and $\rho_{SS}$ constitute a two-parameter family of states, respectively determined by $\langle F\rangle$, $\langle \Phi_d^J\rangle$, and $\langle F^J\rangle$, $\langle \Phi_d^+\rangle$.

Note also that, as a general rule, $G$-invariant operators form algebras [VW01]. The constituent relations for the algebras of unitary symplectic invariant operators will be used throughout the following derivations and read

$$F\Phi_d^J = -\Phi_d^J F = \Phi_d^+F \quad (B.7)$$

for $SS$-invariant states and

$$\Phi_d^+F^J = -\Phi_d^J F^J = F^J\Phi_d^+ \quad (B.8)$$

for $SS$-invariant states.

\(^{3}\)Notice that $(1 \otimes J)\Phi_d^J (1 \otimes J^\dagger) = (J \otimes 1)\Phi_d^+(J^\dagger \otimes 1)$ and therefore the basis operators are symmetric under permutation, as we expect. As a side remark, we note also that since $J$ is real, $J^J = J^T = -J$ (cf. definition (3.26)) and hence $(1 \otimes J)A(1 \otimes J^\dagger) = -(1 \otimes J)A(1 \otimes J)$ for any $A$. We will use this fact frequently, but keep writing $J^J$.
B.2. REPRESENTATION OF THE SETS OF SS AND S\overline{S} INVARIANT STATES

B.2 Representation of the Sets of \( SS \) and \( S\overline{S} \) Invariant States

The set of parameters that describe the family of states \( \rho_{SS} \) and \( \rho_{S\overline{S}} \) define two-dimensional convex sets \( \Sigma \in \text{span}\{\langle F \rangle, \langle \Phi_d \rangle\} \) and \( \hat{\Sigma} \in \text{span}\{\langle F^J \rangle, \langle \Phi_d^+ \rangle\} \). From now on, we focus on the family of \( SS \)-invariant states, but a similar derivation can be done for the \( S\overline{S} \) case. To identify \( \Sigma \) it is convenient to write our states in the orthogonal basis of projectors

\[
\Pi_0 = \Phi_d^+,
\]
\[
\Pi_1 = \frac{1}{2}(\mathbb{1} - F) - \Phi_d^+,
\]
\[
\Pi_2 = \frac{1}{2}(\mathbb{1} + F),
\]

where the relations (B.7-B.8) imply that the previous set indeed defines a projective resolution of the identity:

\[
\Pi_\alpha \Pi_\beta = \delta_{\alpha\beta} \Pi_\beta \quad \text{and} \quad \sum_\alpha \Pi_\alpha = \mathbb{1}.
\]

Projectors (B.9-B.11) correspond to the decomposition of a \( 2n \times 2n \) matrix \( M \) into symplectic trace \( \text{tr}(JM)J \), skew-symmetric symplectic traceless \( (1/2)(M - M^T) - \text{tr}(JM)J \), and symmetric \( (1/2)(M + M^T) \) parts, respectively. We will also need an orthogonal basis for \( \rho_{S\overline{S}} \):

\[
\hat{\Pi}_0 = \Pi_0^\dagger = \Phi_d^+,
\]
\[
\hat{\Pi}_1 = \Pi_1^\dagger = \frac{1}{2}(\mathbb{1} - F^J) - \Phi_d^+,
\]
\[
\hat{\Pi}_2 = \Pi_2^\dagger = \frac{1}{2}(\mathbb{1} + F^J)
\]

with similar properties and \([\Pi_\alpha, \hat{\Pi}_\beta] = 0\).

Any state \( \rho_{SS} \) is then given by the convex combination of the normalized projectors

\[
\rho_{SS} = \sum_i p_i \Pi_\alpha / \text{tr}(\Pi_\alpha)
\]

which correspond to pure quantum states and constitute the extreme points of \( \Sigma \). The coordinates of the vertices of \( \Sigma \) are given by \( (\text{tr}(\Pi_\alpha / \text{tr}(\Pi_\alpha) F), \text{tr}(\Pi_\alpha / \text{tr}(\Pi_\alpha) \Phi_d^J)) \), as plotted in Fig. B.1. In this plot, we also represent the set \( \Sigma^\Gamma \) of operators \( \rho_{SS}^\Gamma \), represented by

\[
\rho_{SS}^\Gamma = \sum_i p_i \hat{\Pi}_\alpha / \text{tr}(\hat{\Pi}_\alpha).
\]

We can already see that this set may contain non-physical operators, as one does not impose positivity nor normalization in (B.17). But the overlap of \( \Sigma \) and \( \Sigma^\Gamma \) has a clear physical meaning: it consists of the region of PPT \( SS \)-invariant states, since here we guarantee that \( \rho_{SS}^\Gamma \) defines a proper quantum state.
Figure B.1: The plot of the set $\Sigma$ of $SS$-invariant states together with the set of partial transposes of $SS$-invariant states $\Sigma^\Gamma$. The set $\Sigma$ is defined by the extreme points $(\langle F \rangle_{\Pi_0}, \langle \Phi \rangle_{\Pi_0}) = (-1, 1)$, $(\langle F \rangle_{\Pi_1}, \langle \Phi \rangle_{\Pi_1}) = (-1, 0)$ and $(\langle F \rangle_{\Pi_2}, \langle \Phi \rangle_{\Pi_2}) = (1, 0)$, considering normalized $\Pi_\alpha$. The set $\Sigma^\Gamma$ has extreme points $(\langle F \rangle_{\hat{\Pi}_0}, \langle \Phi \rangle_{\hat{\Pi}_0}) = (d, -1/d)$, $(\langle F \rangle_{\hat{\Pi}_1}, \langle \Phi \rangle_{\hat{\Pi}_1}) = (0, -1/d)$ and $(\langle F \rangle_{\hat{\Pi}_2}, \langle \Phi \rangle_{\hat{\Pi}_2}) = (0, 1/d)$. The thick line with the arrow represents the partially transposed witness $\tilde{W}_{BH}(p)$. The dashed line represents Werner states; its prolongation to the vertex $(d, -1/d) \equiv \Pi_0^u$ gives NPT isotropic states. An identical plot would be obtained for the case of $SS$-invariance, after the relabeling of axes $\langle F \rangle \to \langle F^J \rangle$ and $\langle \Phi \rangle \to \langle \Phi^J \rangle$. 
B.2. REPRESENTATION OF THE SETS OF SS AND SS INVARIANT STATES

B.2.1 PPT-Bound Entangled and Separable Regions

Let us then focus on the study of the PPT region of states $\rho_{SS}$, resulting from the intersection $\Sigma^d \cap \Sigma$. Elementary planar geometry provide the extreme points $\{\langle F \rangle, \langle \Phi_d \rangle\}$ that define such intersection:

$$r_0 = (0, 0) \quad (B.18)$$
$$r_1 = (0, 1/d) \quad (B.19)$$
$$r_2 = (1, 0) \quad (B.20)$$
$$r_3 = (d/(d + 2), 1/(d + 2)) \quad (B.21)$$

To study the separable region, which obviously lies inside the PPT region, we evaluate the parameters $\langle F \rangle$ and $\langle \Phi_d \rangle$ on a separable state $|uv\rangle$,

$$\langle F \rangle = |\langle uv\rangle|^2 = |\bar{u}_0 v_0 + \bar{u}_1 v_1 + \bar{u}_2 v_2 + \bar{u}_3 v_3 + \cdots + \bar{u}_{2n} v_{2n}|^2,$$
$$\langle \Phi_d \rangle = \frac{1}{d} |\langle J \rangle|^2 = \frac{1}{d} |u_0 v_1 - u_1 v_0 + \cdots + u_{2n-1} v_{2n} - u_{2n} v_{2n-1}|^2 \quad (B.22)$$

where we used the notation $|x\rangle = \sum_i x_i |i\rangle$. From these expressions, we can see that the extreme points $r_0, r_1$ and $r_2$ correspond to separable states, since these expected values can be, respectively, obtained by

$$|uv\rangle_0 = \frac{1}{2\sqrt{2}} (-|0\rangle + |1\rangle + |2\rangle + |3\rangle) \otimes (|0\rangle + |3\rangle) \quad (B.23)$$
$$|uv\rangle_{1,2} = \frac{1}{2} (|0\rangle \mp |1\rangle) \otimes (|0\rangle + |1\rangle) \quad (B.24)$$

With this we prove that any state lying inside the polytope defined by extreme points $\{r_0, r_1, r_2\}$ is separable. In fact, we show that it defines the entire separable region after using the Breuer-Hall map itself. According to it (3.22), a quantum state $\rho$ is entangled if

$$\mathbb{I} \otimes \Lambda_{BH}(\rho) = \frac{1}{d-2} (\text{tr}_B(\rho) \otimes \mathbb{I} - \rho - (\mathbb{I} \otimes J)\rho^T(\mathbb{I} \otimes J^T)) < 0. \quad (B.25)$$

For any SS-invariant state, the reduced state is always the maximally mixed state, $\text{tr}_B \rho_{SS} = \frac{1}{d}$. It follows directly from the fact that this holds for any of its basis operators. Now we multiply the matrix (B.25) by the positive matrix $\Phi_d$, which does not affect the negativity of (B.25). After tracing out, and applying some of the usual relations, we obtain the following relation

$$\text{tr}(\rho \Phi_d^T) > \frac{1 - \text{tr}(\rho F)}{d}. \quad (B.26)$$

This defines a line in the space of parameters above which the states are necessarily entangled. Given that this line contains the extreme points $r_1$ and $r_2$, the region $\{r_0, r_1, r_2\}$ is separable while $\{r_1, r_2, r_3\}$ defines a set of PPT-bound entangled states (see Fig. B.1). This finishes our analysis of unitary symplectic invariant states.
Appendix C

Supporting material for chapter 7

C.1 No-signaling bounds for GYNI inequalities

Here we derive the upper bound $\omega_{ns} \leq 2\omega_c$ for the winning probability $\omega_{ns}$ of no-signalling strategies. We then show that $\omega_{ns} = \omega_c$ for all input distributions $q(x)$ such that $q(x) \leq q(y) = q(\bar{y})$ for some input string $y$. Such distributions include in particular the uniform distribution where all input strings are chosen with equal weight $q(x) = 1/2^N$.

To start we derive the upper bound $\omega_{ns} \leq 2\omega_c$, valid for any distribution $q(x)$. From the definition (7.3), we have that $q(x) \leq \omega_c$ for every input string $x$. This trivially leads to the upper-bound

$$\omega_{ns} \leq \omega_c \sum_x P(a_i = x_{i+1}|x).$$

(C.1)

Notice that this bound is only meaningful when the right-hand side is smaller than 1, since obviously $\omega_{ns} \leq 1$. We now show that for all no-signalling distributions $\sum_x P(a_i = x_{i+1}|x) \leq 2$, from which the bound $\omega_{ns} \leq 2\omega_c$ immediately follows.

First note that from the no-signaling condition,

$$P(a_1, \ldots, a_{k-1}|x_1, \ldots, x_{k-1}) \leq P(a_1, \ldots, a_k|x_1, \ldots, x_k).$$

(C.2)

We now write

$$\sum_x P(a_i = x_{i+1}|x)$$

$$= \sum_{x_1, \ldots, x_N} P(a_1 = x_2, \ldots, a_N = x_1|x_1, \ldots, x_N)$$

$$\leq \sum_{x_1, \ldots, x_N} P(a_1 = x_2, \ldots, a_{N-1} = x_N|x_1, \ldots, x_{N-1})$$

$$= \sum_{x_1, \ldots, x_{N-1}} P(a_1 = x_2, \ldots, a_{N-2} = x_{N-1}|x_1, \ldots, x_{N-2})$$

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where the inequality follows from (C.2) and in the last equality we used the no-signaling condition after summing over \( x_N \). Iteratively performing this last step, we finally obtain

\[
\sum_x P(a_i = x_{i+1} | x) \leq \sum_{x_1, x_2} P(a_1 = x_2 | x_1) \leq 2. \tag{C.3}
\]

We now analyze the no-signaling winning probability for distributions satisfying \( q(x) \leq q(y) = q(\bar{y}) \) for some input string \( y \). Note that for such weights \( \omega_c = q(y) + q(\bar{y}) = 2q(y) \), as easily follows from (7.3). We thus have

\[
\omega_{ns} = \sum_x q(x) P(a_i = x_{i+1} | x) \leq \frac{\omega_c}{2} \sum_x P(a_i = x_{i+1} | x).
\]

But, as we have shown above, \( \sum_x P(a_i = x_{i+1} | x) \leq 2 \) for all no-signalling distributions, and thus \( \omega_{ns} \leq \omega_c \). Since any classical strategy is also a no-signalling strategy, it actually holds that \( \omega_{ns} = \omega_c \).

### C.2 Nonlocal boxes which saturate NS bound for GYNI-3

Here we describe two inequivalent no-signaling correlations which attain \( \omega_{ns} = 4/3 \omega_c \) for the tripartite inequality (7.10). These correlations are extremal non-local boxes in the sense of being vertices of the no-signaling polytope for three parties and binary inputs and outputs [BLM+05].

Writing \((a,b,c)\) for \((a_1,a_2,a_3)\) and \((x,y,z)\) for \((x_1,x_2,x_3)\), we can write the first box as

\[
P_1(a, b, c | x, y, z) = \frac{1}{3} f(a, b, c, x, y, z) \tag{C.4}
\]

where \( f(a, b, c, x, y, z) \) is the boolean function

\[
f(a, b, c, x, y, z) = (1 \oplus b \oplus x \oplus y \oplus xy)(1 \oplus c \oplus z) \\
\quad \oplus a(1 \oplus y \oplus cy \oplus b(c \oplus z)). \tag{C.5}
\]

Similarly, we can write the second box as

\[
P_2(a, b, c | x, y, z) = \frac{2}{3} g(a, b, c, x, y, z) + \frac{1}{3} g'(a, b, c, x, y, z) \tag{C.6}
\]

with \( g \) and \( g' \) the two boolean functions

\[
g(a, b, c, x, y, z) = abc(1 \oplus x)(1 \oplus y)(1 \oplus z) \\
g'(a, b, c, x, y, z) = (1 \oplus a)(1 \oplus b)(1 \oplus c) \\
\quad \oplus xbc \oplus ayb \oplus abz \oplus xyz. \tag{C.7}
\]

Among the boxes that are equivalent to \( P_1 \) under relabeling of parties, inputs, and outputs, a total of 24 of them violate maximally the Bell inequality (7.10), and similarly for 8 of those that are equivalent to \( P_2 \). Even though other tripartite no-signaling boxes (inequivalent to \( P_1 \) or \( P_2 \) under relabeling of parties, inputs, or outputs) violate the Bell inequality (7.10), those 32 boxes obtained from \( P_1 \) and \( P_2 \) are the unique ones that violate it maximally.
C.3 No-signaling bounds for odd and even number of parties

Here we show that for the input distribution (7.9), the no-signaling bound for an even number of parties \( N + 1 \) is always equal to the no-signaling bound for \( N \) parties. Start by considering \( N + 1 \)-GYNI game, where the first \( N \) players use the optimal strategy for the \( N \)-player case and player \( N + 1 \) outputs its input. They then achieve a no-signaling violation equal to the \( N \) case, which imposes the lower bound \( \omega_{ns}(N + 1) \geq \omega_{ns}(N) \). But this is actually the best average success these \( N + 1 \) players can obtain. To see that, consider the game for \( N + 1 \) parties. Allowing players \( N \) and \( N + 1 \) to communicate can only increase the achievable value of \( \omega_{ns}(N + 1) \). Indeed, in this situation the best strategy that player \( N \) can adopt is to output \( x_{N + 1} \), which was communicated to him by player \( N + 1 \), while player \( N + 1 \) needs to guess \( x_1 \) given \( x_N \) and \( x_{N + 1} \). Clearly, the knowledge of \( x_{N + 1} \) is of no use for him since this bit is completely uncorrelated with the rest of the input string. Consequently, the situation is analogous to having players \( 1, \ldots, N - 1, N + 1 \) (i.e. all players except player \( N \)) play the game for \( N \) parties. Therefore \( \omega_{ns}(N + 1) \leq \omega_{ns}(N) \) and we have finally that \( \omega_{ns}(N + 1) = \omega_{ns}(N) \) for odd \( N \).

C.4 Nonlocal computation Inequalities do not define facets

In what follows, we derive a criterion that is necessarily satisfied by any facet-defining Bell inequality associated to the task of nonlocal computation (NLC) [LPSW07], and show that none of the NLC Bell inequalities for boolean functions of two and three input bits are facet-defining.

Nonlocal computation is a distributed task of two parties, where the goal is to compute a given boolean function \( f(z) \) of an \( n \)-bit string \( z \). The input bit string is decomposed into two strings \( x \) and \( y \), such that \( x \oplus y = z \). The bit string \( x \) is sent to party \( A \) while the bit string \( y \) is sent to party \( B \). Upon receiving their input bit strings, \( A \) and \( B \) each output a single bit, \( a \) and \( b \) respectively, such that the following relation holds: \( a \oplus b = f(z) \). Importantly, each party has locally no information about the input bit string \( z \), that is \( P(x_i = z_i) = 1/2 \) for all \( i = 1, \ldots, n \). For each \( n, f(z) \), and distribution of inputs \( \tilde{p}(z) \), we obtain a Bell expression whose value is associated to the probability of success at the task. These NLC inequalities have the form

\[
I(n, f, \tilde{p}) = \sum_z (-1)^{f(z)} \tilde{p}(z) \sum_{x \oplus y = z} (A_x B_y) \leq k(n, f, \tilde{p}) \tag{C.8}
\]

where \( A_x \) and \( B_y \) are observables which take values \( \{-1, 1\} \). Notice that each party measures \( 2^n \) observables.

In Ref. [LPSW07] it is proven that the best classical strategy is given by \( A_x = (-1)^{a_x} \) and \( B_y = (-1)^{b_y} \) with

\[
a_x = u \cdot x, \quad b_y = u \cdot y \oplus \delta, \tag{C.9}
\]

where \( \delta \) denotes a single bit and \( u \) an \( n \)-bit string shared by the parties. This classical strategy, which is a linear approximation of the function \( f \), achieves
a winning probability as high as any quantum resource. Thus the local and quantum bounds of inequalities (C.8) coincide. There exist however no-signaling correlations which can perform with winning probability one at this game.

Checking whether the NLC inequalities (C.8) are facet-defining is in general a hard problem since one should consider any input size \( n \), Boolean function \( f \), and distribution \( \tilde{p}(z) \). Below we give a first simplification to this problem by deriving a necessary criterion satisfied by facet NLC inequalities. Our method is based on a mapping from the \((2, 2^n, 2)\) correlation space — i.e. \((2, 2^n, 2)\) correlation space — in which the NLC inequalities are defined, into the \((n, 2, 2)\) full-correlation space for which the complete set of tight Bell inequalities has been provided in Ref. [WW01a].

To any inequality of the form (C.8) defined by the triple \((n, f, \tilde{p})\), we associate the following Bell inequality in the \((n, 2, 2)\) full-correlation space:

\[
I_{n;22}(n, f, \tilde{p}) = \sum_{\mathbf{z}} c(\mathbf{z}) (C_{z_1} \ldots C_{z_n}) \leq 2^{-n} k(n, f, \tilde{p}) \quad (C.10)
\]

where \( c(\mathbf{z}) = (-1)^{f(\mathbf{z})} \tilde{p}(\mathbf{z}) \), and where we view \( z_i \in \{0, 1\} \) as denoting one of two possible observables \( C_{z_i} \) of party \( i \) taking values \( \{-1, 1\} \) (with \( i = 1, \ldots, n \)).

**Lemma.** If the NLC inequality \( I(n, f, \tilde{p}) \) for \( n \) bits is facet-defining, then the corresponding inequality \( I_{n;22}(n, f, \tilde{p}) \) is facet-defining in the \((n, 2, 2)\) full-correlation space.

**Proof.** The deterministic extremal points of the \((n, 2, 2)\) full-correlation polytope are of the form [WW01a]

\[
(C_{z_1} \ldots C_{z_n}) = (-1)^{u_1 z_1} \ldots (-1)^{u_n z_n} (-1)^{\delta} = (-1)^{\mathbf{u} \cdot \mathbf{z} \oplus \delta} \quad (C.11)
\]

where \( u_i \in \{0, 1\} \) specifies the local strategy of each party and \( \delta \in \{0, 1\} \) represents an additional global sign flip, which we can think of as being carried out by the last party. These deterministic points are thus specified by the single bit \( \delta \) and the \( n \)-bit string \( \mathbf{u} \), and are therefore in one-to-one correspondence with the extremal points (C.9) saturating the inequalities (C.8). For any such strategy specified by \( \delta \) and \( \mathbf{u} \), we have that

\[
\sum_{x \oplus y = z} \langle A_x B_y \rangle = \sum_{x \oplus y = z} (-1)^{\mathbf{u} \cdot (x + y) \oplus \delta} = 2^n (-1)^{\mathbf{u} \cdot \mathbf{z} \oplus \delta} = 2^n \langle C_{z_1} \ldots C_{z_n} \rangle.
\]

It immediately follows from the above identity that the inequalities (C.10) are valid for the \((n, 2, 2)\) full-correlation polytope.

Let us now suppose that the Bell inequality \( I_{n;22}(n, f, \tilde{p}) \leq 2^{-n} k(n, f, \tilde{p}) \) is not facet-defining. Then we can write \( I_{n;22}(n, f, \tilde{p}) = I^1_{n;22}(n, f, \tilde{p}) + I^2_{n;22}(n, f, \tilde{p}) \) and \( k(n, f, \tilde{p}) = k^1(n, f, \tilde{p}) + k^2(n, f, \tilde{p}) \) for some \( I^1_{n;22}(n, f, \tilde{p}) \), \( I^2_{n;22}(n, f, \tilde{p}) \), \( k^1(n, f, \tilde{p}) \), and \( k^2(n, f, \tilde{p}) \) such that

\[
I^1_{n;22}(n, f, \tilde{p}) \leq 2^{-n} k^1(n, f, \tilde{p}) \quad (C.13)
\]

and

\[
I^2_{n;22}(n, f, \tilde{p}) \leq 2^{-n} k^2(n, f, \tilde{p}) \quad (C.14)
\]
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are valid inequalities for the \((n, 2, 2)\) full-correlation polytope, i.e., they are satisfied by all deterministic points of the form (C.11). But then it follows from the above correspondence between deterministic point of the \((n, 2, 2)\) polytope and the \((2, 2^n, 2)\) polytope that 

\[ I(n, f, \bar{p}) = I^1(n, f, \bar{p}) + I^2(n, f, \bar{p}), \]

where

\[ I^1(n, f, \bar{p}) \leq k^1(n, f, \bar{p}) \]

and

\[ I^2(n, f, \bar{p}) \leq k^2(n, f, \bar{p}) \]

are valid NLC inequalities. This implies that \(I(n, f, \bar{p}) \leq k(n, f, \bar{p})\) is not facet-defining for the \((2, 2^n, 2)\) polytope, from which the statement of the Lemma follows. □

The above Lemma implies that it is sufficient to restrict our analysis on NLC inequalities associated with facet inequalities in the \((n, 2, 2)\)-full correlation space. In Ref. [WW01a] a construction for the coefficients \(c(z)\) of all facet \((n, 2, 2)\) correlation Bell inequalities has been given. For small number of inputs, i.e. \(n = 2, 3\), we have explicitly verified that none of the corresponding NLC inequalities are facet-defining; all these inequalities can actually be expressed as sums of CHSH inequalities. For larger \(n\) however, a similar analysis becomes difficult due to the large number of facet \((n, 2, 2)\) Bell inequalities and the high dimensionality of the \((2, 2^n, 2)\) correlation space.

*Note added.* While completing this thesis, Winter (with contributions from Brunner, Skrzypczyk and myself) proved that in fact none of the NLC inequalities define facets of the local polytope.
Resumen

La teoría cuántica es el modelo físico que describe con más precisión los fenómenos microscópicos. A pesar de ello, su formulación y predicciones contra-intuitivas, han provocado la difícil implantación de esta teoría en la comunidad científica. El modelo cuántico abandona principios tradicionalmente considerados esenciales en cualquier teoría de la Naturaleza, consiguiendo así ceñir sus predicciones a las observaciones experimentales en las que la teoría clásica se había mostrado incorrecta. Por ejemplo, la Física Cuántica abandona el determinismo: en general, los observables no tienen un valor definido antes de ser medidos. Además, existen observables cuyos valores no se pueden determinar simultáneamente con total precisión.

Existen otras intrigantes características propias de la teoría cuántica como por ejemplo: (i) el entrelazamiento cuántico, o la existencia de estados puros cuánticos que no pueden ser factorizados en estados producto; (ii) la existencia de correlaciones no-locales entre los resultados de medidas locales efectuadas en sistemas separados que comparten un estado cuántico entrelazado. Se piensa que estos fenómenos constituyen una nueva forma de estudiar la teoría cuántica. De hecho, hace dos décadas, los aspectos menos asimilados de la teoría empezaron a ser explotados como una fuente de recursos muy útiles para tareas de procesamiento de información. Estos recursos permiten llevar a cabo tareas imposibles con métodos clásicos, como por ejemplo, la criptografía (cuántica) perfectamente segura [BB84, Eke91], la teleportación (cuántica) perfecta [BBC+93] o la factorización en números primos en tiempo polinomial [Sho97]. Y ésta es apenas una pequeña muestra de las posibilidades ofrecidas por la codificación de información en sistemas cuánticos. [NC00, HHHH09]. La Teoría de la Información Cuántica surgió para estudiar el nuevo campo interdisciplinar entre la Física y la Teoría de la Información.

A pesar de todo lo que ha logrado la Teoría de la Información Cuántica, aún no ha conseguido contestar a la pregunta básica: Por qué los recursos cuánticos ofrecen ventaja sobre los clásicos? Además, aún se siguen buscando métodos eficientes para la identificación, caracterización y cuantificación de dichos recursos. Por otro lado, este punto de vista operacional ha permitido obtener otra perspectiva de las bases de la Física Cuántica y es factible que ésto posibilite, finalmente, substituir los axiomas estándar de la Física Cuántica por principios intuitivos.
D.1 Cuestiones abordadas y Resultados

La principal motivación de esta tesis doctoral es contribuir a la identificación y caracterización del entrelazamiento y no-localidad cuánticos. El procedimiento general consiste en comparar estos recursos con los recursos clásicos y supercuánticos, además de explotar el régimen de múltiples partes. Se espera con ello profundizar en la estructura de la Teoría Cuántica. A continuación, se enumeran las principales cuestiones abordadas en esta tesis y los principales resultados obtenidos.

**Como simplificar los métodos experimentales de detección de estados entrelazados.** El entrelazamiento es utilizado en la gran mayoría de protocolos cuánticos, siendo así importante que sea detectado de forma eficiente a nivel experimental. Desafortunadamente, el mejor criterio para la identificación de estados entrelazados no puede ser implementado directamente en el laboratorio. Esto ocurre debido a que el criterio utiliza mapas positivos, que describen operaciones no implementables en práctica. Sin embargo, es posible aproximar la acción de estos mapas sobre estados cuánticos por medio de aproximaciones físicas estructurales.

Consideramos canales cuánticos que resultan de la aproximación física estructural aplicada a mapas positivos. En todos los casos estudiados, siempre que el mapa positivo es óptimo, observamos que estos canales destruyen el entrelazamiento, en el sentido de que son inútiles para su distribución. Esta propiedad permite que la acción de estos canales pueda ser sustituida por un procedimiento de medida y preparación de estados cuánticos. Por tanto, la aproximación física de los mapas positivos óptimos tiene una implementación particularmente sencilla, que puede ser utilizada en métodos directos de detección de estados entrelazados más simples y eficientes que los actuales.

**Robustez de la no-localidad en presencia de ruido.** La no-localidad cuántica es un también un recurso necesario para ciertas tareas. Se sabe además que la presencia de entrelazamiento es necesaria pero no suficiente para la no-localidad. Es entonces relevante definir los estados entrelazados que carecen de contenido no-local.

Considerando el escenario bipartido, estudiamos la relación entre no-localidad y entrelazamiento, así como la resistencia de las correlaciones cuánticas no-locales al ruido. Demostramos que para una clase de estados mixtos suficientemente general, el entrelazamiento no garantiza la existencia de no-localidad. En particular, encontramos límites a la cantidad de ruido que transforma estados entrelazados no-locales en estados entrelazados locales.

**Métodos para la caracterización de correlaciones cuánticas no-locales multipartidas.** Las correlaciones compartidas entre más de dos partes de un sistema se llaman multipartidas. Aunque este escenario sea rico en cantidad y variedad de correlaciones que pueden surgir, su potencial permanece aún prácticamente inexplorado.

En este trabajo, utilizamos el formalismo multipartido para estudiar correlaciones no-locales, tanto bipartidas como multipartidas. Utilizaremos también
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una estrategia de medidas locales en algunas partes del sistema y, posteriormente, estudiamos las propiedades no-locales del estado de las demás partes. Conseguimos de este modo demostrar que existen estados cuánticos presentando no-localidad multipartida, lo que representa la forma más pura de no-localidad en sistemas que pueden ser divididos en varios subsistemas. Además, esta característica se puede observar tanto en estados puros como en estados mezcla.

Establecemos también una conexión directa entre no-localidad cuántica y destilación unidireccional de entrelazamiento.

Activación de correlaciones no-locales La posibilidad de activar no-localidad tomando diversas copias de un estado que no presenta este recurso, es muy relevante tanto desde un punto de vista práctico como conceptual. Utilizando nuestras técnicas para el estudio de la no-localidad en el escenario multipartido, demosntramos que la no-localidad puede de hecho ser activada. Damos los primeros ejemplos de estados cuánticos, cuyo contenido no-local es activado por medio de medidas colectivas en varias copias. Esto ocurre tanto para la no-localidad bipartida así como para la multipartida.

Los límites de la no-localidad cuántica multipartida. Por último, consideramos una forma distinta de caracterizar correlaciones cuánticas. Estamos interesados en conocer cuales son sus límites en las tareas que pueden ser realizadas usando recursos cuánticos y como podríamos llevarlas a cabo si tuviéramos acceso a correlaciones supra-cuánticas, pero que respeten el principio de no transmisión instantánea de información. Este punto de vista es particularmente interesante ya que no se conoce el motivo por el cual las correlaciones cuánticas no presentan una mayor no-localidad.

Definimos el primer juego multipartido para el cual jugadores clásicos y cuánticos son igual de efectivos, pero pueden ser batidos por jugadores supra-cuánticos. Nuestros resultados sugieren que correlaciones físicas multipartidas obedecen a un principio de no-signalización generalizado.
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