

## REVIEW ARTICLE

# Spatiotemporal optical solitons

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Received 1 December 2004, accepted for publication 1 February 2005

Published 15 April 2005

Online at [stacks.iop.org/JOptB/7/R53](http://stacks.iop.org/JOptB/7/R53)

## Abstract

In the course of the past several years, a new level of understanding has been achieved about conditions for the existence, stability, and generation of spatiotemporal optical solitons, which are nondiffracting and nondispersing wavepackets propagating in nonlinear optical media. Experimentally, effectively two-dimensional (2D) spatiotemporal solitons that overcome diffraction in one transverse spatial dimension have been created in quadratic nonlinear media. With regard to the theory, fundamentally new features of light pulses that self-trap in one or two transverse spatial dimensions and do not spread out in time, when propagating in various optical media, were thoroughly investigated in models with various nonlinearities. Stable vorticity-carrying spatiotemporal solitons have been predicted too, in media with competing nonlinearities (quadratic–cubic or cubic–quintic). This article offers an up-to-date survey of experimental and theoretical results in this field. Both achievements and outstanding difficulties are reviewed, and open problems are highlighted. Also briefly described are recent predictions for stable 2D and 3D solitons in Bose–Einstein condensates supported by full or low-dimensional optical lattices.

**Keywords:** Bose–Einstein condensation, collapse, dispersion management, Gross–Pitaevskii equation, group-velocity dispersion, light bullet, matter-wave soliton, modulational instability, nonlinear optics, nonlinear Schrödinger equation, optical lattice, optical vortex, second-harmonic generation, self-focusing, solitary wave, spatial soliton, spatiotemporal soliton, variational approximation, vortex soliton

(Some figures in this article are in colour only in the electronic version)

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## List of acronyms used in the text

1D	one dimensional	CQ	cubic–quintic (nonlinearity)
2D	two dimensional	FF	fundamental frequency
3D	three dimensional	GVD	group-velocity dispersion
BC	boundary conditions	GVM	group-velocity mismatch
BEC	Bose–Einstein condensation	LB	light bullet
BG	Bragg grating	MI	modulational instability
		NLS	nonlinear Schrödinger (equation)
		OL	optical lattice

SH	second harmonic
SHG	second-harmonic generation
SIT	self-induced transparency
STS	spatiotemporal soliton
VA	variational approximation
VK	Vakhitov–Kolokolov (stability criterion)

## 1. Introduction: basic concepts and terminology

A general property of electromagnetic wavepackets is that they tend to spread out as they evolve. A fundamental cause for this is that distinct frequency components, which are superposed to create the wavepacket, propagate with different velocities and/or in different directions. An example is the transverse spreading of a laser beam due to diffraction. Similarly, light pulses spread in time as they propagate in a material medium, as, due to the group-velocity dispersion (GVD), each Fourier component of the pulse has a different velocity. These examples pertain to linear propagation of beams or pulses. Nonlinear effects generally accelerate the disintegration of a wavepacket. However, under special conditions, nonlinearity may compensate the linear effects. The resulting balanced localized pulse or beam of light, that propagates without decay, is generally known as a *soliton* or, more properly, a solitary wave (sometimes, the term ‘soliton’ is reserved for pulses in exactly integrable nonlinear-wave models). Thus, optical solitons are localized electromagnetic waves that propagate stably in nonlinear media with dispersion, diffraction or both.

Temporal solitons in single-mode optical fibres are the prototypical optical solitons; these were predicted in 1973 (Hasegawa and Tappert 1973), and first observed experimentally in 1980 (Mollenauer *et al* 1980). The formation of such temporal solitons can be understood intuitively as follows. In an optical fibre (or any other suitable optical medium), the effective index of refraction  $n$  depends on the intensity  $I$ , so that  $n = n_0 + \Delta n(I)$ , where, in the first approximation, the small nonlinear correction is proportional to the intensity,  $\Delta n(I) = n_2 I$ . Usual materials feature *self-focusing*, which implies  $n_2 > 0$ , and instantaneous nonlinear response of the medium (the latter means that there is no temporal delay between  $\Delta n(I)$  and  $I$ ); such materials are referred to as *Kerr media*. In a more general case, the medium is self-focusing or self-defocusing in the case of, respectively,  $d(\Delta n)/dI > 0$  or  $d(\Delta n)/dI < 0$ .

In the course of the propagation, the pulse accumulates a phase shift that, due to the intensity-dependent correction  $n_2 I$  to the refractive index, mimics the temporal shape of the pulse,  $I = I(t)$ . To understand this in a more accurate form, one may start from the normalized wave equation for the electric field  $E$ ,

$$E_{zz} - (n^2 E)_{tt} = 0, \quad (1)$$

where  $z$  is the propagation distance,  $t$  is time, and  $n$  is the above-mentioned refractive index (detailed derivation of the wave equation can be found, e.g., in the book by Agrawal (1995)). A solution to equation (1) is looked for as

$$E(z, t) = u(z)e^{ik_0 z - i\omega_0 t} + u^*(z)e^{-ik_0 z + i\omega_0 t}, \quad (2)$$

where  $\exp(ik_0 z - i\omega_0 t)$  represents a rapidly varying carrier wave, and  $u(z, t)$  is a slowly varying local amplitude.

Substituting this in equation (1), in the lowest approximation one obtains a dispersion relation between the propagation constant (wavenumber)  $k$  and the frequency  $\omega$ ,  $k_0^2 = (n_0 \omega_0)^2$ , and the next-order approximation yields an evolution equation for the amplitude,

$$i \frac{du}{dz} + \frac{n_0 n_2}{k_0} \omega_0^2 I u = 0 \quad (3)$$

(actually,  $I = |u|^2$ , as the intensity is tantamount to the squared amplitude). A solution to equation (3) is simply  $\Delta\phi = (n_0 n_2)(\omega_0^2/k_0) I z$ , where  $\Delta\phi$  is a nonlinear shift of the wave’s phase. Because the corresponding frequency shift is  $\Delta\omega = -\partial\Delta\phi/\partial t$ , one obtains

$$\Delta\omega = -\frac{n_0 n_2}{k_0} \omega_0^2 \frac{dI}{dt} z. \quad (4)$$

It follows from equation (4) that the lower-frequency components of the pulse, with  $\Delta\omega < 0$ , develop near its front, where  $dI/dt > 0$ , while higher frequencies, with  $\Delta\omega > 0$ , develop close to the rear of the pulse, where  $dI/dt < 0$ .

Actually, the dielectric response of the material medium is not instantaneous, but is characterized by a finite temporal delay. This implies that the linear part,  $\epsilon \equiv n_0^2$ , of the multiplier  $n^2$  in the wave equation (1) (the dynamic dielectric permeability) is actually a linear operator, rather than simply a multiplier. The accordingly modified form of the linear term becomes  $(\int_0^\infty \epsilon(\tau) E(t - \tau) d\tau)_{tt}$ , where  $\tau$  is the delay time. Finally, approximating this nonlocal-in-time expression by a quasi-local expansion,  $\epsilon_0 E_{tt} + \epsilon_2 E_{tttt} + \dots$ , gives rise to second- and higher-order GVD terms in the eventual propagation equation, which can be translated into the corresponding linear dispersion relation,  $k = k(\omega)$  (Agrawal 1995).

In particular, the normal GVD (which means that waves with a higher frequency have a smaller group velocity, as conveniently expressed by the condition  $\beta_2 \equiv d^2 k/d\omega^2 > 0$ ) would reinforce the above trend to the temporal separation between the low- and high-frequency components of the pulse, contributing to its rapid spread. In contrast, anomalous GVD ( $\beta_2 < 0$ ), which occurs too in real materials, may compensate the nonlinearity-induced spreading. With the magnitudes of the dispersion and intensity properly matched, the balance may be perfect, giving rise to very robust pulses, that are referred to as (1 + 1)-dimensional ((1 + 1)D, for brevity) solitons, where the first ‘1’ corresponds to  $t$ , and the second ‘1’ relates to  $z$ . Extensive research has led to the development of soliton-based telecommunications systems (Hasegawa and Kodama 1995, Iannone *et al* 1998). The first commercial fibre-optic telecommunications link using the solitons (actually, they are *dispersion-managed* solitons, which implies that the local dispersion coefficient  $\beta_2(z)$  periodically alternates between positive and negative values), about 3000 km long, was launched in Australia in 2003 (McEntee 2003). While this is the first direct commercial application of the soliton concept, we note that related phenomena helped to achieve other important technological advances over the years, such as self-focusing which led to Kerr-lens mode locking for the generation of short laser pulses (see, e.g., a paper by Haus (2000)).

*Spatial solitons* originate from a balance of self-focusing and diffraction. In this case, a laser beam experiences the

Kerr-induced nonlinear contribution to the index of refraction that follows the spatial (rather than temporal) profile of the intensity, and thus acts as an effective lens. In one transverse dimension, the diffraction formally resembles the anomalous dispersion in the temporal domain (see above); therefore the intensity-dependent lens can exactly compensate diffraction, and the resulting beam may propagate without spreading or self-compression. Such a beam has the form of a stripe in a planar waveguide (the waveguide confines the beam in the other transverse direction), as was first observed by Aitchison *et al* (1990). In view of the above-mentioned mathematical similarity of the one-dimensional diffraction to anomalous chromatic dispersion, the existence of soliton solutions (which may also be regarded as (1 + 1)D objects, where, this time, the first '1' refers to the transverse spatial coordinate,  $x$ , rather than  $t$  in the case of temporal solitons) in this case is not surprising. However, (2 + 1)D spatial solitons (self-formed cylindrical beams) in media with the Kerr (cubic, or  $\chi^{(3)}$ ) nonlinearity are unstable, unlike their (1 + 1)D counterparts, because two-dimensional fluctuations may destroy the balance between the nonlinearity and diffraction in that case. In particular, an increase of the intensity leads to self-focusing of the cylindrical beam, which further increases the intensity and the corresponding intensity-dependent correction to the index of refraction, which leads to still stronger focusing and increase of the intensity, and so on. This self-accelerating process of the nonlinear self-focusing is referred to as *collapse* of the beam (see, e.g., the review by Bergé 1998). In an idealized mathematical model of the Kerr medium, the collapse would result in formation of a singularity after passing a finite propagation distance, but in reality new physical phenomena enter into the play, such as higher-order nonlinearities, non-paraxiality, and material damage. One way to avoid such a behaviour and stabilize the (2 + 1)D spatial soliton is to have *saturation* of the nonlinearity: if the nonlinear coefficient  $n_2$  itself depends on the intensity  $I$ , so that it starts to *decrease* with  $I$ , the self-focusing may reach a *stable* equilibrium with diffraction.

Studies of spatial solitons have made rapid progress since the mid-1990s, when two new soliton-supporting nonlinear optical interactions became available to experiments (see Stegeman *et al* 2000). First, Segev *et al* (1992) had predicted that the photorefractive effect in electro-optic materials could be exploited to create a saturable nonlinear index of refraction that would support solitons. Photorefractive solitons were observed experimentally soon afterwards (Duree *et al* 1993), and since then a variety of such solitons, of both 1D and 2D types, have been discovered and explored (Segev and Christodoulides 2002). This includes landmark advances, such as self-trapping of incoherent light (Mitchel and Segev 1997) and the recent generation of ordinary (Fleischer *et al* 2003) and vortex (Fleischer *et al* 2004, Neshev *et al* 2004) solitons in optically induced photonic lattices, as well as robust necklace-shaped soliton clusters (Yang *et al* 2005); the lattice solitons in photorefractive media were first predicted by Efremidis *et al* (2002).

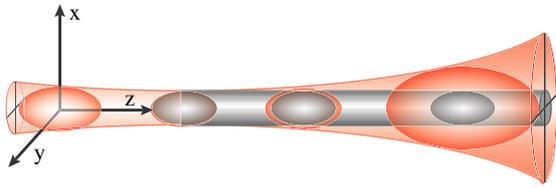
Second, concomitant with the work on photorefractive materials, there was a resurgence of interest in the study of effects produced by the parametric interaction of two or three waves in quadratically nonlinear ( $\chi^{(2)}$ ) media. In appropriate

limits, the parametric interactions lead to a so-called cascading mechanism, which induces an effective approximately cubic ( $\chi^{(3)}$ ) nonlinearity through a superposition of two  $\chi^{(2)}$  processes (the resulting effective  $\chi^{(3)}$  nonlinearity may be either self-focusing or self-defocusing, corresponding to  $\Delta n = n_2 I$  with  $n_2 > 0$  and  $n_2 < 0$ , respectively). This process was identified theoretically in the 1960s, but only sporadic experimental results appeared (Thomas and Taran 1972, Belashenkov *et al* 1989) until DeSalvo *et al* (1992) reported a systematic study of the nonlinear phase shift produced in the second-harmonic generation (SHG), and Stegeman *et al* (1993) properly recognized its potential to impress large nonlinear phase shifts on light beams and signals (for a review, see Stegeman *et al* 1996). This interaction becomes efficient close to the point of phase matching between the fundamental-frequency (FF) and second-harmonic fields involved in the SHG interaction (where the phase velocities of the two waves coincide). In this case, the cascading limit does not hold, and genuine 'quadratic' parametric interactions occur.

The fact that quadratic nonlinearities feature an effective dynamic saturation at large intensities can be readily understood by noticing that the higher the intensity in one field, the stronger the tendency to energy conversion to the other field. Hence, the exchange of energy between the two fields, and the sensitivity of the energy exchange between the FF and SH fields to the relative phase between them, are the key ingredients of the physical picture that may stabilize the pulse or beam in this case. Thus, the presence of the two fields is a fundamental difference from the Kerr media, which entails new experimental complications, but also opens new opportunities. Extensive work for solitons in  $\chi^{(2)}$  media has been performed during the last decade (see reviews by Etrich *et al* (2000), Buryak *et al* (2002), Torner and Barthelemy (2003)). (2 + 1)D and (1 + 1)D spatial solitons in quadratic media were first generated experimentally by, respectively, Torruellas *et al* (1995) and Schiek *et al* (1996).

Another important example of a (2 + 1)D spatial soliton is the *optical vortex*, which is a soliton of the *dark* type, in the form of a 'hole' in an extended background, which is supported by a vortex phase pattern imprinted onto the background (Swartzlander and Law 1992, Snyder *et al* 1992, Tikhonenko *et al* 1995, Duree *et al* 1995, Chen *et al* 1997, Kivshar *et al* 1998). This pattern is characterized by the phase circulation of  $2\pi S$ , with an integer  $S$ , around the hole. Fundamental vortices (with  $S = 1$ ) are stable in the case of a self-defocusing  $\chi^{(3)}$  nonlinearity, while all the higher-order vortices with  $S \geq 2$  are predicted to be unstable, splitting into a set of fundamental ones. However, in  $\chi^{(2)}$  media all the vortices are unstable, as their spatially uniform background is subject to modulational instability. Nevertheless, some complex vortex structures, not yet fully analysed theoretically, were observed in a  $\chi^{(2)}$  medium by Di Trapani *et al* (2000), in cases when the instability growth was effectively inhibited, under conditions that, through the above-mentioned cascading mechanism, the quadratic nonlinearity mimics a self-defocusing cubic nonlinearity for one of the waves involved in the  $\chi^{(2)}$  process.

One of the major goals in the study of nonlinear wave interactions in optics is the generation of pulses that are localized in all the transverse dimensions of space, as well as in time (for a moving soliton, the latter is equivalent to



**Figure 1.** Illustration of the formation of a spatiotemporal soliton due to the simultaneous balance of diffraction and dispersion by nonlinear self-focusing.

localization in the longitudinal coordinate)—*spatiotemporal* solitons (STS). These are  $(2 + 1 + 1)$ D objects, where the ‘2’ is the transverse dimension, the first ‘1’ stands for the temporal variable, and the last ‘1’ pertains to the propagation coordinate. The search for the formation of such objects dates back to the early days of the field. In particular, such pulses in Kerr media were considered by Silberberg (1990), who had coined the term ‘light bullets’ (LB) for them, which stresses their particle-like nature (a ‘light bullet’ propagating in a planar waveguide would be a  $(1 + 1 + 1)$ D object). In contrast to the extensive developments in the studies of temporal and spatial solitons in one and two dimensions, experimental progress toward the production of STS in the three-dimensional case has been slow. To date,  $(2 + 1 + 1)$ D STS have not yet been observed.

The formation of  $(2 + 1 + 1)$ D solitons involves the same conditions as for their  $(2 + 1)$ D and  $(1 + 1)$ D lower-dimensional counterparts, which provide for the proper balance between the linear spreading and a suitable nonlinearity. Thus, STS may be understood as the result of the *simultaneous* balance of diffraction and GVD by the transverse self-focusing and nonlinear phase modulation in the longitudinal direction, respectively (figure 1). Hence, one might naively expect that nonlinear responses suitable for the formation of a stable low-dimensional soliton would also be adequate for the formation of solitons in all dimensions. That such is not the case, and, actually, that dimensionality is a central issue in the formation of solitons (first of all, because of the problem of stability against the collapse in the multidimensional case), was learned in the early days of the work in the field, as illustrated above for the case of the Kerr nonlinearity. Actually, it turns out that an *effective* strength of a given nonlinearity critically depends on the dimensionality of the physical setting where the nonlinearity acts (see, e.g., Kuznetsov *et al* 1986). Therefore, progress in the area requires specific conceptual, analytical, numerical, and experimental advances. This review aims to summarize these advances along with some important open questions.

In summary, the quest for spatiotemporal solitons, or light bullets, faces two main challenges: first, physically relevant models of nonlinear optical systems, based on evolution equations that allow *stable* three-dimensional propagation, ought to be identified; second, suitable materials should be found where such models can be implemented. Thus, advances in both theoretical and experimental directions are necessary, making it appropriate to review progress in the two areas. We will begin with a summary of experimental results, that will illustrate the main issues. Recent theoretical work, which both underlies the experiments and treats a variety of phenomena that are still beyond current experimental capabilities, will

be summarized after that. In the presentation of theoretical models and results, we chiefly focus on those models which are closest to current experiments, i.e., the ones with the  $\chi^{(2)}$  nonlinearity, and on issues which play the key role in the experiments. We will also give a brief overview of theoretical predictions for two- and three-dimensional solitons in Bose–Einstein condensates (BECs), in the case of the self-attractive cubic nonlinearity, where the stabilization of the solitons is provided for by optical lattices created in the BEC. These objects, being very different physically from the optical STSs, nevertheless have much in common with them, as regards the mathematical models and understanding of the underlying dynamics. A brief discussion of open problems is included at the end of the review.

## 2. Experimental advances

### 2.1. Cubic nonlinear media

The following are requisites for the generation of stable completely localized (i.e., finite-energy) STS in homogeneous media: self-focusing nonlinearity, anomalous GVD, and one or more processes that can prevent collapse of the pulse. It is useful to define the characteristic lengths for dispersion,  $L_{DS} = \tau^2/|\beta_2|$ , and diffraction,  $L_{DF} = \pi\rho^2/\lambda$ , where  $\rho$  is the radius of the beam, and  $\lambda$  is the wavelength of the carrying electromagnetic wave. These are the distances over which a pulse/beam begins to broaden appreciably in time and space, respectively, if the propagation is linear. Stable passage of several characteristic lengths is required to identify a soliton in the experiment. In a medium with positive Kerr nonlinearity and anomalous GVD, a pulse will simultaneously compress in time and space (which is the beginning of the collapse) if  $L_{DS} \sim L_{DF}$ . Effects that are not included in the simple model, such as multiphoton absorption and stimulated Raman scattering, can prevent the collapse from occurring. This expectation is borne out by recent experiments. In particular, Eisenberg *et al* (2001) investigated the propagation of intense femtosecond-duration pulses in a planar fused-silica waveguide. The wavelength was chosen so that the material had anomalous GVD, and the above-mentioned condition,  $L_{DS} \sim L_{DF}$  (as concerns the diffraction in the waveguide’s plane), held. Space–time self-focusing *without collapse* was observed for a range of input intensities (however, no soliton was formed). Numerical simulations show that multiphoton ionization and stimulated Raman scattering indeed arrest the onset of collapse in such a situation. These experiments provided the first direct observation of space–time focusing in a Kerr medium, but the resulting pulses are not (and cannot be) solitons—the dissipative processes that arrest the collapse deplete the pulse’s energy, and eventually cause it to spread out in the transverse and longitudinal (i.e., spatial and temporal) directions.

Besides the above-mentioned dissipative processes, collapse may be arrested by a variety of mechanisms that conserve energy: non-paraxiality, higher-order dispersion (Fibich *et al* 2002), or self-generation of nonlocal nonlinearities mediated by the *rectification*, i.e., generation of a slowly varying zero-harmonic field (Torres *et al* 2002a, 2002b, Crasovan *et al* 2003b, Leblond 1998a, 1998b), to

name a few. These effects may lead to new forms of soliton-like propagation. In particular, Goorjian and Silberberg (1997) observed stable spatiotemporal propagation of few-cycle pulses over many characteristic lengths in a numerical solution of the full system of the Maxwell's equations in a nonlinear medium.

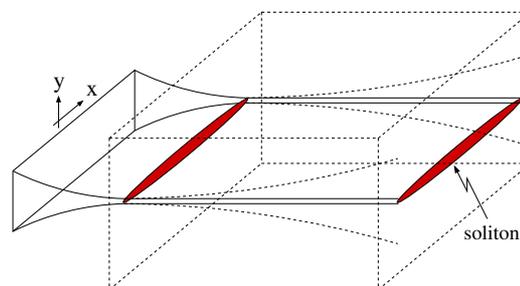
On the other hand, Towers and Malomed (2002) had shown that complete stabilization of a cylindrical (2 + 1)D spatial soliton can be secured in a layered medium with 'nonlinearity management', i.e., periodic alternation of the sign of the Kerr nonlinearity along the propagation distance. It is interesting that the same mechanism supports stable 2D solitons in the temporal domain—not in optics, but rather in BECs, where the periodic change of the sign of nonlinearity is provided by the so-called Feshbach resonance (Saito and Ueda 2003, Abdullaev *et al* 2003, Montesinos *et al* 2004b), including the case of two-component BECs (Montesinos *et al* 2004a). However, the model with the alternating nonlinearity sign *cannot* provide for stabilization of full 3D solitons.

A somewhat related theoretical result, which will be presented in a brief form below, predicts stabilization of pulsating STSs in a (1 + 1 + 1)D medium combining the cubic nonlinearity and the above-mentioned dispersion management, i.e., periodic alternation of the sign of the GVD coefficient as a function of the propagation coordinate  $z$  (Matuszewski *et al* 2004). In this case too, stabilization is not possible in the full (2 + 1 + 1)D case.

## 2.2. Quadratic nonlinear media

Following the pioneering work by Karamzin and Sukhorukov (1976), as early as in 1981 it was shown that solutions for multidimensional solitons, including the full (2 + 1 + 1)D STS, exist and are stable in  $\chi^{(2)}$  media (Kanashov and Rubenchik 1981). Malomed *et al* (1997) reported the first treatment of STS in quadratic media based on the variational approach (VA), and delineated the conditions under which STS would be stable. Some details of the VA will be discussed below; it is important to mention that the theoretical prediction of STS in this case motivated the experiment.

Several of the experimental issues that must be addressed in the study of  $\chi^{(2)}$  media are consequences of the two-field nature of the problem, which involves the FF and SH components. A major impediment to the experimental realization of STS is the requirement of anomalous GVD at both harmonics, of a magnitude large enough that the dispersion lengths be small compared to the length of available quadratic crystals (in fact, there is a possibility to observe long-lived soliton-like propagation in the case when the GVD is slightly normal at SH, as was shown by Torner (1999), and by Towers *et al* (2003), and will be discussed in some detail below). The latter is usually limited to a few centimetres, which implies that the dispersion length must be below 1 cm. In many materials it is not possible to choose the carrier wavelength so that the FF and SH simultaneously experience anomalous GVD and negligible absorption. Typically, acceptably transparent materials only have large anomalous GVD at wavelengths in the 2–4  $\mu\text{m}$  range, where sources of well-controlled high-energy femtosecond-duration pulses, needed to launch nonlinear

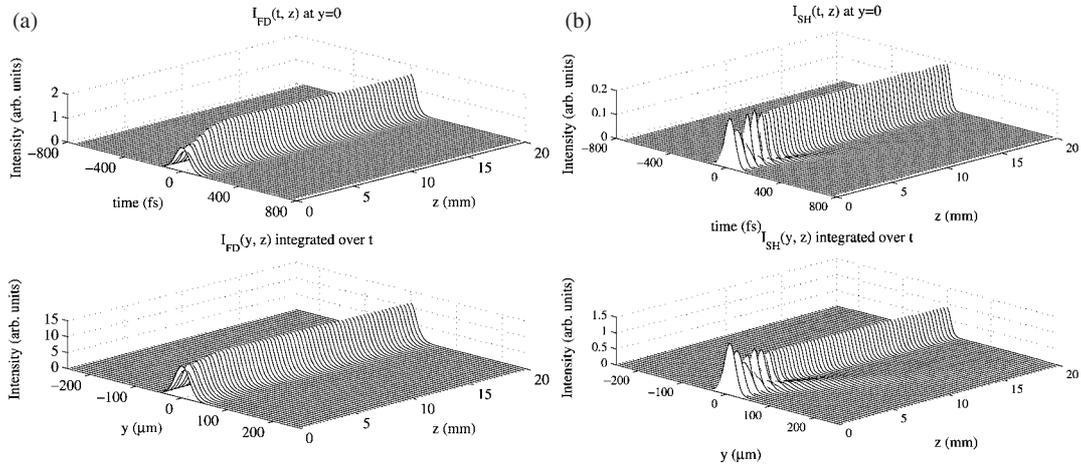


**Figure 2.** Schematic diagram of the experimental arrangement used to observe (1 + 1 + 1)D spatiotemporal solitons.

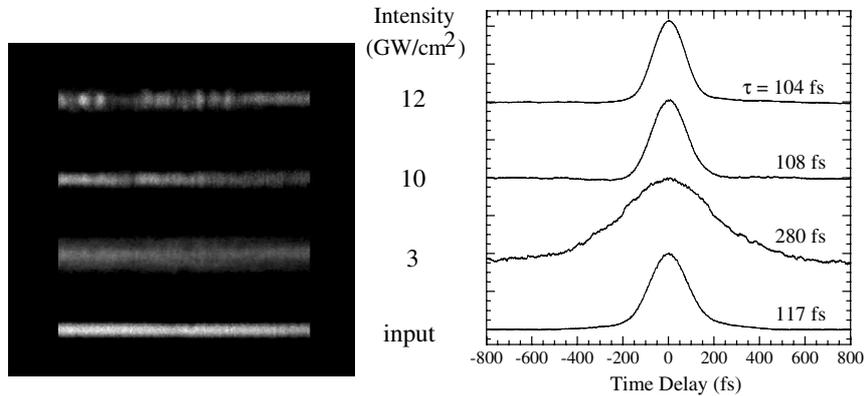
propagation, perform poorly. A closely related issue is that, generally, the group velocities of pulses at the FF and SH differ substantially; the resulting group-velocity mismatch (GVM) significantly restricts the range of parameters over which solitons may form. In the linear propagation, the FF and SH pulses would separate temporally at a characteristic GVM propagation length ( $L_{\text{GVM}} = c\tau/(n_{1g} - n_{2g})$ , with  $c$  the velocity of light,  $\tau$  the pulse duration, and  $n_{1g}$  and  $n_{2g}$  the group indices at the FF and SH, respectively), which tends to weaken the coupling between the harmonics and naturally counters soliton formation. The solitons that form despite the presence of small GVM are referred to as 'walking' ones, because they actually propagate at some velocity falling within the interval between the group velocities of the FF and SH fields, i.e., they 'walk' in the reference frame that moves at the FF group velocity (Torner *et al* 1996, Etrich *et al* 1997). However, the temporal group-velocity mismatch prevents spatial trapping near phase matching for pulses that are too short in the temporal dimension—typically the limit is a few picoseconds (Carrasco *et al* 2001a). The latter feature was directly confirmed in the experiment (Pioger *et al* 2002).

GVD and GVM can effectively be controlled by tilting the amplitude fronts of a pulse with respect to the phase fronts. The tilt is impressed on a pulse through angular dispersion, most commonly by the use of prisms or diffraction gratings, and allows desired values of GVD and GVM to be obtained at wavelengths convenient to the experiment, such as 800 nm or 1  $\mu\text{m}$ . The ability to control both GVD and GVM with tilted pulse fronts was exploited in the first experimental observation of temporal solitons in  $\chi^{(2)}$  media, by Di Trapani *et al* (1998). In that case, the tilt was used to create effective anomalous GVD for pulses launched into a barium metaborate (BBO) crystal.

Liu and co-workers used tilted pulses to produce (1 + 1 + 1)D STS (Liu *et al* 1999, 2000a). Pulses of 120 fs duration at 800 nm were converted into a narrow stripe by means of cylindrical optics, as shown in figure 2. The pulse tilt was imposed in the  $x$ -direction, and the beam was focused in the  $y$ -direction onto a crystal of lithium iodate or BBO. In lithium iodate the tilt could be arranged so that the GVM was close to zero, while the propagation could be monitored over three dispersion/diffraction lengths. A pulse launched with suitable phase mismatch and intensity evolved to a *stable spatiotemporal profile*, and was observed at the exit face of the crystal, compressed in both time and space. Roughly 10% of the input energy was radiated away, owing to the fact that the launched pulse contained the FF only and had thus to rearrange



**Figure 3.** Numerical simulations that show the formation of a (1 + 1 + 1)D spatiotemporal soliton. Temporal (upper panels) and spatial (lower panels) profiles of the fundamental (left) and harmonic (right) are shown.



**Figure 4.** Experimental observations of (1 + 1 + 1)D spatiotemporal solitons and modulational instability of larger pulses. The spatial profiles on the left and temporal profiles on the right were recorded with the corresponding input intensities listed in the centre of the figure. At low intensity (3 GW cm<sup>-2</sup>), the beam diffracts in the vertical direction and the pulse disperses in time. At 10 GW cm<sup>-2</sup>, the field at the output face of the crystal (~5 characteristic lengths) supports its compressed shape in time and space and is very close to the launched field. At 12 GW cm<sup>-2</sup> (when the pulse’s energy is larger than it must be in the soliton), the beam breaks into elliptical filaments, which gradually evolve into circular ones. The temporal profile is that of a single filament.

itself into the two-wave soliton. These pulses were the first experimentally observed optical spatiotemporal solitons, as well as the first temporal solitons in  $\chi^{(2)}$  media to be directly observed over several characteristic lengths.

Walking STS were studied in experiments with BBO, in which the material dispersion and birefringence allowed  $L_{DS}/L_{GVM} \sim 3$  (Liu *et al* 2000a). This relation implies that, in the linear propagation, the FF and SH pulses would move apart by  $\sim 3$  times the initial pulse duration while passing one dispersion length. The large GVM reduces the acceptance range for the soliton formation; experimentally, solitons could only be observed with the phase mismatch greater than a threshold value, in agreement with the theory (Carrasco *et al* 2001b). With the phase mismatch near the optimum value, as determined by numerical simulations (see figure 3), stable spatiotemporal propagation was observed over five characteristic lengths. The measured spatial and temporal profiles are shown in figure 4. A series of similar experiments with varying phase mismatch and intensity allowed Liu *et al* to delineate threshold conditions for the STS formation, which agree well with numerical simulations.

One of the limitations to soliton formation is modulation instability (MI). Only the (2 + 1 + 1)D STS is truly a stable object; all lower-dimensional solitons in the 3D medium are subject to MI (Kanashov and Rubenchik 1981), and hence they tend to break up in time and/or space. Starting from the conditions that produce STS, an increase of the input intensity by  $\sim 20\%$  causes the stripe to spontaneously break into filaments along the major axis, see figure 4 (Liu *et al* 2000b). Higher intensities produce denser filaments, which is a clear signature of the MI. The filaments are initially elliptical, but quickly make their cross section circular, while maintaining  $\sim 100$  fs temporal duration. Numerical simulations show that it should eventually be possible to generate true LBs through this instability of the (1 + 1 + 1)D STS. However, the angular dispersion used to control GVD and GVM in the experiments of Liu *et al* caused the filaments to disperse soon after their formation, thus precluding the generation of fully confined STS.

Finally, we briefly mention that (1 + 1 + 1)D STS can also be generated through a non-collinear interaction of two input pulses. Through mutual nonlinear trapping of the input

FF fields and the SH field, STS form and propagate along the bisector of the angle between the input beams (Liu *et al* 2000c). This Y-geometry was originally suggested by Drummond *et al* (1999), and naturally lends itself to implementation of the logical AND operation: a pulse is detected along the bisector only if both input pulses are present.

The use of tilted pulse fronts is an undesirable feature of all STS produced experimentally to date, as it precludes the generation of (2 + 1 + 1)D light bullets. Thus, new strategies for the potential generation of STS without pulse tilt are required. Some recent theoretical efforts in this direction will be described below.

### 3. Basic theoretical models

#### 3.1. Single-component models

It is well known that, within the framework of the standard approach based on slowly varying amplitudes of the electromagnetic field and paraxial approximation for diffraction, the evolution of the local field amplitude  $u$  in a dispersive multidimensional nonlinear medium (which is introduced similarly to equation (2)) is governed by the generalized nonlinear Schrödinger (NLS) equation (Newell and Moloney 1992, Akhmediev and Ankiewicz 1997, Kivshar and Agrawal 2003):

$$iu_z + (1/2)(\nabla_{\perp}^2 u + \sigma u_{\tau\tau}) + f(|u|^2)u = 0; \quad (5)$$

cf equation (3). Here, as above,  $z$  is the coordinate along the propagation direction of the carrier wave,  $\tau \equiv t - z/V_{\text{gr}}$  is the ‘reduced time’, with  $V_{\text{gr}}$  being the group velocity of the carrier wave (the reduced time has the same meaning as in fibre optics (Agrawal 1995)),  $\sigma$  is the GVD coefficient, which depends on the wavelength of the carrier wave (actually, it is tantamount to  $-\beta_2$  which was defined above),  $\nabla_{\perp}^2$  is the Laplacian (diffraction operator) acting on the transverse coordinates  $(x, y)$ , and the function  $f(|u|^2)$  describes a nonlinear correction to the effective refractive index of the material medium. As explained above in the context of equation (3), the classical Kerr effect corresponds to  $f(|u|^2) = K|u|^2$ , where  $K$  is a constant. We denote the cases with the self-focusing and self-defocusing cubic nonlinearity, i.e.,  $K > 0$  and  $K < 0$ , as  $\chi_+^{(3)}$  and  $\chi_-^{(3)}$ , respectively. The above-mentioned model of the medium composed of alternating self-focusing and self-defocusing layers, which supports stable (2+1)D spatial solitons, corresponds to  $K(z)$  periodically jumping between positive and negative values (Towers and Malomed 2002). In accordance with what was also described above, the cases with  $\sigma > 0$  and  $\sigma < 0$  in equation (5) are referred to as anomalous and normal GVD. In what follows below, we will only consider the case of the anomalous dispersion; therefore, if  $\sigma$  is constant, we may always set use the normalization  $\sigma \equiv +1$  (in the single-component model). On the other hand, the dispersion management (in the multidimensional medium) corresponds to  $\sigma(z)$  periodically varying between positive and negative values (Matuszewski *et al* 2004, 2005). Even if  $\sigma$  and the coefficients of the nonlinearity,  $f(|u|^2)$ , are functions of  $z$ , equation (5) conserves two dynamical invariants, namely, the

norm (which is the power, in the spatial-domain models, and energy, in the spatiotemporal-domain ones), and momentum,

$$E = \int |u|^2 \, \mathbf{dr}, \quad \mathbf{P} = \frac{i}{2} \int (u \nabla u^* - u^* \nabla u) \, \mathbf{dr}. \quad (6)$$

Note that these expressions do not depend on the form of  $f(|u|^2)$  in equation (5).

The first issue that needs to be taken care of in the theoretical approach to STS is the above-mentioned wave collapse (spontaneous formation of a singularity): solitary-wave solutions have a chance to be stable in a uniform medium only if the nonlinearity adopted in the model does not lead to collapse, in a given spatial dimension. In particular, LBs supported by the usual  $\chi_+^{(3)}$  nonlinearity are unstable due to this; therefore the simple spatially uniform Kerr nonlinearity is not an appropriate setting for the creation of STS.

More accurately, the axially symmetric soliton in the 2D NLS equation (it is commonly called a *Townes soliton*, which was, as a matter of fact, the first example of a soliton considered in nonlinear optics; see the seminal paper by Chiao *et al* (1964)) is *weakly unstable*: no eigenmode of small perturbations around the Townes soliton is an exponentially growing one, but there is an algebraically growing mode, corresponding to nonlinear instability. In contrast to this, spherically symmetric solitons in the 3D NLS equation are strongly (linearly) unstable in the ordinary sense, featuring an unstable eigenvalue in the spectrum of perturbation eigenmodes. Accordingly, the collapse is *weak* in the 2D NLS equation: to initiate the collapse, the energy  $E$  of the initial pulse (see equation (6)) must exceed a minimum (threshold) value, which is exactly the energy of the Townes soliton; in the 3D case, the collapse has no threshold.

On the other hand, LBs in the  $\chi_+^{(3)}$  medium may be stabilized by creating a waveguiding channel in it, as was shown numerically by Raghavan and Agrawal (2000). The channel is a cylindrical region with an increased value of the refractive index. The possibility of the stabilization of LBs in a channel is not surprising, as the usual temporal soliton in an optical fibre may be considered as a ‘light bullet’ propagating in a strongly confining waveguide, as was stressed by Manassah *et al* (1988).

The simplest possibility for producing a stable STS in a uniform medium is to choose a saturable nonlinearity, with  $f(|u|^2) = -(1 + |u|^2/u_0^2)^{-1}$  in equation (5), where  $u_0$  is a constant. Indeed, simulations performed by Edmundson and Enns (1992) and Enns and Rangnekar (1992) had shown that (2 + 1 + 1)D solitons are stable in this case. In fact, LBs supported by saturable nonlinearity are bistable objects (two different stable solitons may have equal energies), which makes it also possible to consider switching between them (Enns *et al* 1992). However, while some physical settings, for instance vapours of alkali metals, may be modelled by a saturable nonlinearity, and soliton-like objects were experimentally observed in them (Tikhonenko *et al* 1996), these media exhibit strong dissipation too.

As was shown numerically by Quiroga-Teixeiro *et al* (1999), and Desyatnikov *et al* (2000) (see further references below), another model that can generate stable higher-dimensional solitons is the one with  $\chi_+^{(3)}$ :  $\chi_-^{(5)}$ , alias cubic–quintic (CQ), nonlinearity. In this case, the function  $f(|u|^2)$  in

equation (5) combines a self-focusing term,  $+|u|^2$ , and a self-defocusing one,  $-|u|^4$ . Coefficients in front of these terms can be normalized so that equation (5) takes a parameter-free form,

$$iu_z + (1/2)(\nabla_{\perp}^2 u + u_{\tau\tau}) + (|u|^2 - |u|^4)u = 0. \quad (7)$$

A nonlinearity of the CQ type was derived in some simplified (truncated) models, e.g., those combining interaction of light with two-level atoms and dipole interactions between the atoms (Bowden *et al* 1991), or the same light–matter interaction and Kerr nonlinearity of waveguides (Aközbeke and John 1998). Recently, experimental observations of a nonlinear dielectric response in chalcogenide glasses (Smektala *et al* 2000, Boudebs *et al* 2003, Ogusu *et al* 2004) and in organic materials (Zhan *et al* 2002), which might be modelled by the CQ terms, in combination with additional ones (including those accounting for the two-photon (nonlinear) absorption), were reported. The real applicability of the CQ model to these newly investigated media requires additional experimental explorations. In particular, a recent analysis by Chen *et al* (2004) shows that the nonlinear loss in the available chalcogenide glasses leaves open a window of opportunity for observing (1+1+1)D STS, as well as (2+1)D spatial solitons, but *not* the fully localized (2+1+1)D ‘light bullets’.

Lastly, it is relevant to mention that multidimensional Bose–Einstein condensates (BECs) obey (with a very high accuracy) the Gross–Pitaevskii (GP) equation, whose normalized form is (Dalfovo *et al* 1999):

$$iu_t + (1/2)\nabla^2 u + g|u|^2 u + U(\mathbf{r})u = 0, \quad (8)$$

where  $u(t, \mathbf{r})$  is the normalized single-atom wavefunction,  $g = +1$  and  $-1$  for the attracting and repelling atoms, respectively, and  $U(\mathbf{r})$  is the external potential (in this case, the evolution variable is time, rather than  $z$ ). In the case of the BEC, the nonlinearity is always cubic. Experiments with BEC always demand the presence of a parabolic trap, corresponding to  $U = (1/2)\Omega^2 r^2$ , with a constant  $\Omega$ . For the prediction of stable multidimensional solitons, crucially important is another potential term, which accounts for a periodic optical lattice (OL), that can be easily built in the experiment as an interference pattern induced by counter-propagating coherent laser beams illuminating the condensate. In the 3D case, the OL potential is

$$U(\mathbf{r}) = -\epsilon[\cos(kx) + \cos(ky) + \cos(kz)], \quad (9)$$

with constant  $\epsilon > 0$  and  $k$  (the sign minus in front of  $\epsilon$  implies that  $x = y = z = 0$  is a local minimum of the potential). The 2D case corresponds to the expression (9) without the term  $\cos(kz)$ . An interesting possibility is also presented by the *low-dimensional potential*, i.e., the 2D potential in the 3D case, and the 1D potential in the 2D case (Baizakov *et al* 2004a, 2004b). The low-dimensional potential is quite relevant to the experiment, simply because it is easier to create it by means of the laser beams.

### 3.2. Two-wave models with the quadratic nonlinearity

As explained in the previous section, the only media in which STS have been already observed experimentally are optical crystals featuring the  $\chi^{(2)}$  nonlinearity; hence it is necessary to

give a review of main theoretical results obtained in respective models. Equations governing the evolution of amplitudes  $u$  and  $v$  of the FF and SH fields in dispersive  $\chi^{(2)}$  bulk media, including the  $\chi^{(3)}$  nonlinearity too, are well known (Menyuk *et al* 1994, Karpierz 1995, Buryak *et al* 1995, Bang *et al* 1998, Etrich *et al* 2000, Buryak *et al* 2002)

$$iu_z + (1/2)(\nabla_{\perp}^2 u + u_{\tau\tau}) + u^* v - \Gamma(|u|^2 + 2|v|^2)u = 0, \quad (10)$$

$$i(v_z + cv_{\tau}) + (1/4)(\nabla_{\perp}^2 v + \delta \cdot u_{\tau\tau}) - \beta v + u^2 - 2\Gamma(2|u|^2 + |v|^2)v = 0. \quad (11)$$

Here,  $c$  and  $\beta$  are differences (mismatches) of the group and phase velocities between the FF and SH waves (the former was called GVM in the previous section),  $\Gamma$  is the  $\chi^{(3)}$  coefficient (in this notation,  $\Gamma > 0$  corresponds to the self-defocusing,  $\chi^{(3)}$ , cubic nonlinearity), while the  $\chi^{(2)}$  coefficient is normalized to be 1, and  $\delta$  is the ratio of the GVD coefficients at SH and FF. Note that the GVD coefficient in equation (10) is set to be +1, which implies that the temporal dispersion at the FF is anomalous; cf equation (5) (otherwise, there is no chance to produce STS). In fact, equations (10) and (11) present the simplest model for the light propagation in media with *competing nonlinearities*, the  $\chi^{(2)}$  part of the model corresponding to the *type-I* interaction, which involves only one polarization of light.

Equations describing planar  $\chi^{(2)}$  waveguides differ from the above ones in that  $\nabla_{\perp}^2$  is replaced by  $\partial^2/\partial x^2$ , where  $x$  is the single transverse coordinate. As for the (2+1)D spatial solitons, equations generating them differ from the above ones by dropping the  $\tau$ -derivatives. Note that, on replacing the temporal variable  $\tau$  by the second transverse coordinate  $y$ , any single-component model for (1+1+1)D solitons in a planar waveguide with anomalous GVD is made formally similar to a model for (2+1)D spatial solitons in a bulk medium with the same nonlinearity; therefore solitary-wave solutions may be interpreted in either way. However, the spatiotemporal model is exactly equivalent to the spatial one in the bulk medium solely if  $\delta = 1$  in equation (11) (the case of a ‘spatiotemporal isotropy’; see also below); otherwise the spatiotemporal model is a more general one.

Equations (10) and (11) conserve the energy (or ‘number of quanta’, also called the Manley–Rowe invariant),

$$E = \int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{+\infty} dx (|u|^2 + |v|^2), \quad \text{or} \\ E = \int_{-\infty}^{+\infty} d\tau \int_0^{+\infty} 2\pi r dr (|u|^2 + |v|^2), \quad (12)$$

in the (1+1+1)D and (2+1+1) cases, respectively. Besides the energy, other conserved quantities are the three components of the momentum (cf equation (12)), the Hamiltonian  $H$ , and, in the absence of the walk-off term, the  $z$ -component  $L$  of the orbital angular momentum too (Akhmediev and Ankiewicz 1997).

### 3.3. Other models

A combination of the  $\chi^{(2)}$  nonlinearity and linear dispersion induced by the resonant reflection on a Bragg grating (BG) in a 3D medium gives rise to another collapse-free model which, in principle, can support STS, as was shown by He and Drummond (1998). The corresponding model is based

on a system of equations for the amplitudes  $u_{\pm}$  and  $v_{\pm}$  of the right- and left-travelling FF and SH waves,

$$\begin{aligned} i(C^{-1}\partial_t - \partial_x + \nabla_{\perp}^2)u_+ + u_+^*v_+ + \lambda u_- &= 0, \\ i(C^{-1}\partial_t + \partial_x + \nabla_{\perp}^2)u_- + u_-^*v_- + \lambda u_+ &= 0, \\ 2i(\partial_t - \partial_x + \nabla_{\perp}^2 - \beta)v_+ + u_+^2/2 + v_- &= 0, \\ 2i(\partial_t + \partial_x + \nabla_{\perp}^2 - \beta)v_- + u_-^2/2 + v_+ &= 0. \end{aligned} \quad (13)$$

Here,  $C$  and  $\lambda$  are the group velocity and Bragg reflectivity of the FF wave, while the same coefficients for SH are normalized to be 1, and  $\beta$  is the phase mismatch; cf equations (10) and (11). Effectively stable STS were found in simulations of this model. However, one should take into regard that the linear spectrum of this system contains no true bandgap, the presence of which is a necessary condition for the existence of solitons in the rigorous sense. Therefore, the STS found in this system may actually have tiny nonvanishing ‘tail’, similarly to the STS found by Towers *et al* (2003) in the  $\chi^{(2)}$  model with relatively weak *normal* GVD at the SH.

STS were also studied theoretically in other systems, such as models of off-resonance two-level media (Mel’nikov *et al* 2000), and the Maxwell–Bloch equations describing self-induced transparency (SIT) in a multidimensional medium,

$$-i\nabla_{\perp}^2 E + n^2 E_{\tau} + E_z + i(1 - n^2)E - P = 0, \quad (14)$$

$$P_{\tau} - EW = 0, \quad W_{\tau} + (1/2)(E^*P + P^*E) = 0, \quad (15)$$

where  $E$  and  $P$  are the complex electric field and polarization of the medium, real  $W$  is the population inversion of the two-level atoms, and  $n$  is the refractive index. Equations (14) and (15) do not give rise to collapse; therefore LBs may exist and be stable in this model, as was shown numerically by Blaauboer *et al* (2000). Besides that, stable LBs in the SIT medium exist in a guiding channel (Blaauboer *et al* 2002). The channel can be created by giving the refractive index  $n$  (see equation (14)) a graded profile in the radial direction, similar to the above-mentioned one, proposed by Raghavan and Agrawal (2000) to stabilize LBs in the  $\chi_+^{(3)}$  medium.

Coming back to the  $\chi^{(2)}$  nonlinearity, another potential strategy for achieving formation of fully three-dimensional LBs was suggested by Torner *et al* (2001). Their scheme is based on the concept of *tandem structures*, which are composed of periodically alternating linear dispersive and  $\chi^{(2)}$  nonlinear layers, so that the nonlinearity and GVD are provided by different materials. In this scheme, the STS are composed of guiding-centre wavepackets that ought to periodically pass pieces with high quadratic nonlinearity but no GVD, followed by linear pieces with GVD only. Numerical simulations show that long-lived STS can form in a variety of tandem structures with feasible domain lengths (Torner *et al* 2001). A step forward along this direction has been reported recently by Carrasco *et al* (2004), who experimentally observed self-focusing of light in tandems with periodically varying Poynting vector walk-off.

#### 4. Main theoretical results

As was mentioned above, theoretical results for LBs in media with various nonlinearities are, currently, far more

advanced than experimental observations. The objective of this section is to summarize the most essential theoretical results, which should suggest perspectives for further advances in the experimental studies.

##### 4.1. Solitary-wave solutions

A solution to the 3D NLS equation (5) (with the GVD parameter  $\sigma = 1$ ) describing a stationary STS (generally speaking, a ‘spinning’ one) is sought in the form

$$u(z, r, \theta, \tau) = U(r, \tau) \exp(i\kappa z + iS\theta), \quad (16)$$

where  $r$  and  $\theta$  are the polar coordinates in the transverse plane, real  $\kappa$  is a wavenumber,  $S = 0, 1, 2, \dots$  is the vorticity (‘spin’), and the real function  $U(r, \tau)$  obeys the following boundary conditions (BC):

$$U(r, \tau) \sim \xi^{-(D-1)/2} \exp(-\sqrt{2\kappa}\xi), \quad \xi^2 \equiv r^2 + \tau^2 \quad (17)$$

for  $r, |\tau| \rightarrow \infty$  ( $D = 2$  and  $3$  correspond to the  $(1 + 1 + 1)$ D and  $(2 + 1 + 1)$ D cases, respectively; in the former case,  $\xi^2 \equiv x^2 + \tau^2$ ), and  $U \sim r^{|S|}$  when  $r \rightarrow 0$  (if  $S = 0$ , the BC at  $r = 0$  is  $U_r(r = 0) = 0$ ). A corollary of equation (17) is that solitons may only exist with  $\kappa > 0$ .

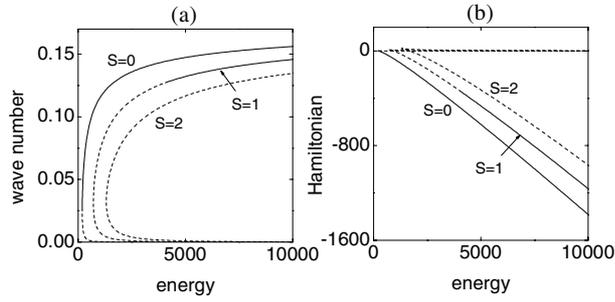
The form (16) implies that the spin (if  $S \neq 0$ ) is aligned with the propagation direction  $z$ . Solutions of a more general type would be extremely complex; therefore they have never been considered (in a system of two nonlinearly coupled 3D *discrete* NLS equations with the cubic nonlinearity, a stable complex of two *orthogonal* vortices was recently found by Kevrekidis *et al* (2004b); that model has no interpretation in terms of the guided-wave propagation, but applies to BECs and lattices of coupled resonant microcavities).

Similar to equation (16), a stationary-soliton solution to equations (10) and (11) is sought for as

$$\begin{aligned} u(z, r, \theta, \tau) &= U(r, \theta) \exp(i\kappa z + iS\theta), \\ v(z, r, \theta, \tau) &= V(r, \theta) \exp(2i\kappa z + 2iS\theta) \end{aligned} \quad (18)$$

(the functions  $U$  and  $V$  may be assumed real if the GVM parameter  $c$  is zero). These stationary solutions constitute one-parameter families of solitons, with the wavenumber shift  $\kappa$  acting as an intrinsic parameter of the family (in the presence of the walk-off, the families of stationary walking solitons are characterized by a second parameter, the soliton’s velocity). The stationary STS realize extrema of the Hamiltonian  $H$  at fixed energy  $E$ :  $\delta_{\mathbb{F}}(H + \kappa E) = 0$ , where  $\delta_{\mathbb{F}}$  stands for the functional, or Fréchet, derivative.

These STS exist for values of wavenumber shifts  $\kappa$  such that the soliton is not in resonance with the linear dispersive waves, to avoid energy leakage from the soliton. This condition confines an existence domain of the solitons. Once this domain was found, stability of solitons is a crucial issue. In the case of single-parameter (non-walking) families of solitons, a linear-stability border is determined by the Vakhitov–Kolokolov (VK) criterion,  $dE/d\kappa = 0$  (Vakhitov and Kolokolov 1973). This result can be understood in terms of simple geometrical arguments, recalling that, as was said above, the stationary solitary wave solutions realize a local extremum (maximum or minimum) of  $H$  for given energy  $E$  (see, e.g., Kuznetsov



**Figure 5.** The wavenumber  $\kappa$  (a) and Hamiltonian  $H$  (b) of the 3D solitons in media with focusing cubic and defocusing quintic nonlinearities versus their energy  $E$ .

*et al* 1986, Akhmediev and Ankiewicz 1997, Torner *et al* 1995a). The lower branch of the  $E$ – $H$  diagram (where the  $E$ – $H$  curve is concave) may represent a stable soliton family, while the solitons situated on the upper branch (where the  $E$ – $H$  curve is convex) are all definitely unstable. When the lowest-order soliton solutions realize an absolute minimum of  $H$  for given  $E$ , then they are stable (for a recent overview of the use of Hamiltonian-versus-energy curves in the analysis of the existence and stability of solitons in conservative systems, see Ankiewicz and Akhmediev (2003)). Typical examples of  $\kappa = \kappa(E)$  and  $H = H(E)$  plots are shown in figure 5 for the case of 3D solitons in media with focusing cubic and defocusing quintic nonlinearities (Mihalache *et al* 2002a); the full and dashed lines in this figure correspond to stable and unstable solitons (for a detailed discussion, see below the section devoted to spinning solitons).

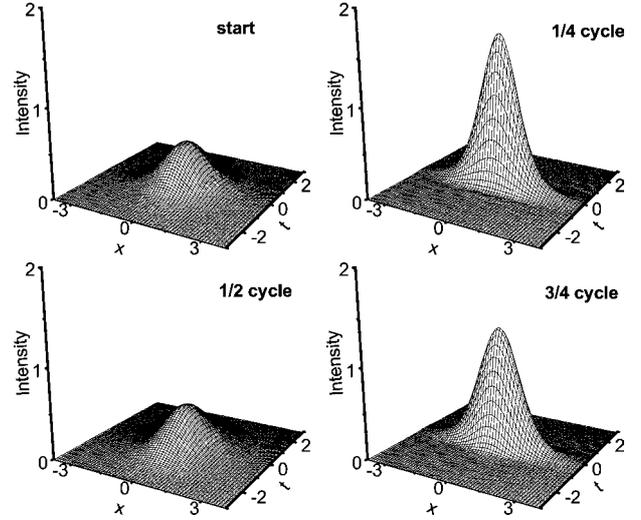
It should be stressed that the VK criterion does not provide information about possible oscillatory instability, accounted for by pairs or quartets of complex-conjugate eigenvalues (see, for example, Akhmediev *et al* 1993, De Rossi *et al* 1998, Mihalache *et al* 1998b, Schöllmann *et al* 1999). Therefore, this criterion has to be used with caution when applied to stationary solutions that do not realize the absolute minimum of  $H$  for given  $E$ , for example those corresponding to spinning solitons.

#### 4.2. Two- and three-dimensional solitons in models with the cubic nonlinearity

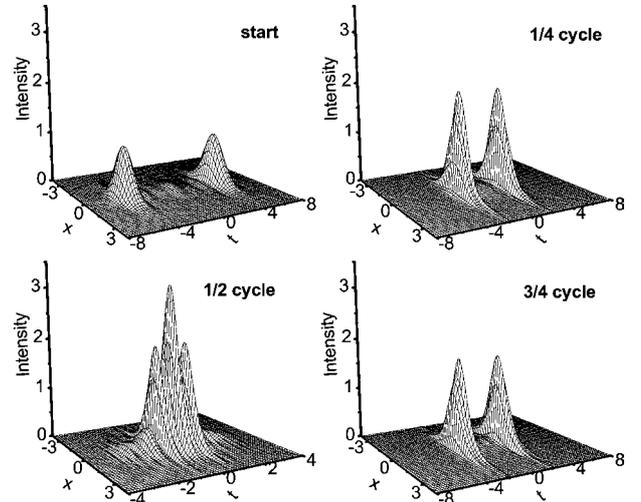
Before proceeding to non-Kerr nonlinearities, it is relevant to briefly describe the possibility of stabilization of  $(1 + 1 + 1)$ D STSs in the dispersion-management  $\chi^{(3)}$  model, which amounts to equation (5) with  $\sigma = \sigma(z)$ ,  $f(|u|^2) \equiv |u|^2$ , and *one* transverse coordinate  $x$ . Matuszewski *et al* (2004) have demonstrated this possibility using semi-analytical and numerical methods. The analytical approach is based on the VA (variational approximation), with the following ansatz for the soliton:

$$u(z, x, \tau) = A(z) \exp \left\{ i\phi(z) - \frac{1}{2} \left[ \frac{x^2}{W^2(z)} + \frac{\tau^2}{T^2(z)} \right] + \frac{i}{2} [b(z)x^2 + \beta(z)\tau^2] \right\}, \quad (19)$$

where  $A$  and  $\phi$  are the amplitude and phase of the soliton,  $W$  and  $T$  are its transverse and temporal widths, and  $b$  and  $\beta$  are the spatial and temporal chirps. All these variational parameters are real. The application of the VA (as described



**Figure 6.** A series of snapshots, taken from direct simulations of the  $(1 + 1 + 1)$ D equation (5) with the dispersion management, which illustrate stable oscillations of the pulse deep inside its stability region.



**Figure 7.** The same as figure 6, but for a pulse taken close to a border of the stability region. In this case, the spatiotemporal soliton periodically splits into two subpulses that recombine again.

in the review by Malomed (2002)) leads to the following evolution equations for the parameters:

$$b = W'/W, \quad \beta = \sigma^{-1}T'/T, \quad (20)$$

$$W'' = \frac{1}{W^3} - \frac{E}{2W^2T}, \quad (21)$$

$$T'' - \frac{\sigma'}{\sigma}T' = \frac{\sigma^2}{T^3} - \frac{\sigma E}{2WT^2}, \quad (22)$$

where the prime stands for  $d/dz$ , and  $E$  is the soliton's energy. Numerical solution of these equations, followed by direct simulations of the underlying NLS equation, have made it possible to identify a sufficiently large stability region in the soliton's parameter space. Direct simulations of equation (5) confirm the stability of the pulsating solitons in that region. A typical example of such a soliton is shown in figure 6. The oscillations produce no tangible radiation loss. More

accurate examination of the numerical data shows that some amount of radiation is emitted from the soliton when it is passing a layer with the normal GVD, but this radiation wave is *reabsorbed* by the soliton within the anomalous-GVD layer. An interesting situation occurs close to borders of the stability region: in that case, the pulse periodically splits and recombines, remaining a stable state; see figure 7. However, in the 3D case (corresponding to equation (5) with two transverse coordinates) neither the VA nor direct simulations can produce any stable soliton in the dispersion-managed model.

Another possibility for stabilizing both 2D and 3D solitons in the NLS equation with the cubic self-focusing nonlinearity is provided by the OL potential (see equation (9)), as was demonstrated by Baizakov *et al* (2003, 2004a, 2004b). In the 2D case, the stationary solution may be approximated by the ansatz

$$u = A \exp[-i\mu t - (ax^2 + by^2)]. \quad (23)$$

With the full OL potential, which involves both  $\cos(kx)$  and  $\cos(ky)$  in equation (9), one can set  $a = b$ , while with the quasi-1D OL, which involves only  $\cos(kx)$ , the variational parameters  $a$  and  $b$  must be kept independent in the formulas given below (the OL wavenumber is normalized to be  $k = 2\sqrt{2}$ , as in the works by Baizakov *et al* (2003, 2004a, 2004b)). The final prediction of the VA is that, with the full (two-dimensional) OL, the 2D solitons exist and are stable, pursuant to the VK criterion, in an interval

$$8\pi(1 - 8e^{-2}\epsilon) < E < 8\pi, \quad (24)$$

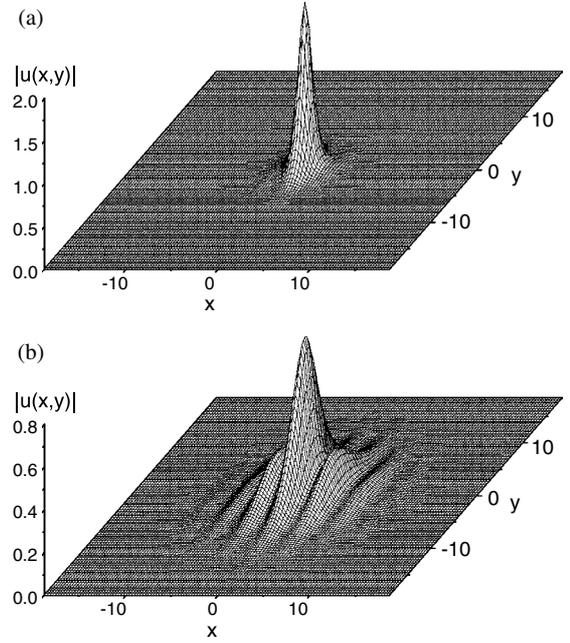
and with the quasi-1D lattice, their existence and stability interval is

$$8\pi\sqrt{1 - 8e^{-2}\epsilon} < E < 8\pi, \quad (25)$$

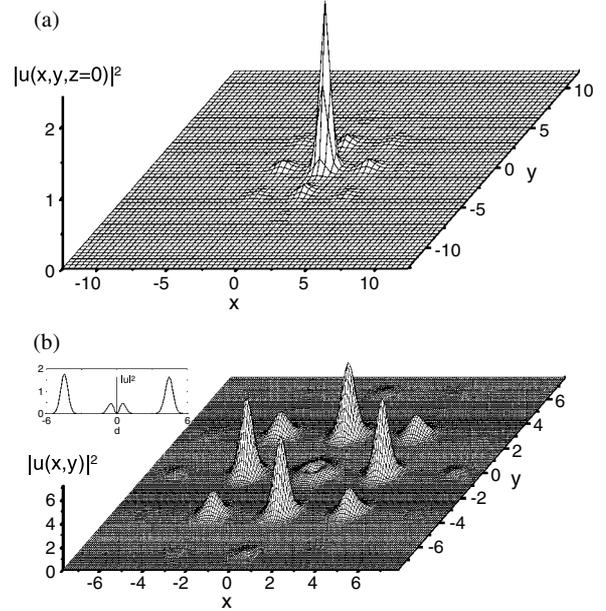
assuming  $\epsilon < e^2/8 \approx 0.924$  (the norm  $E$ , defined as per equation (6), has the meaning of the number of atoms in the condensate, rather than the energy in the optical models). The purport of these results is that, in the absence of the OL ( $\epsilon = 0$ ), both intervals (24) and (25) shrink to a single point,  $E = 8\pi$ , which is nothing but the VA prediction for the norm of the Townes soliton (a numerically found value of its norm is 23.4; i.e., the VA predicts it with an error of  $\approx 7\%$ ). Thus, the single value of the norm of the weakly unstable (neutrally stable, in terms of its eigenvalue spectrum) Townes soliton in the free 2D space is stretched by the OL into a finite interval, (24) or (25), in which the emerging soliton family is stable.

Direct numerical simulations corroborate these predictions quite well. Figure 8 shows examples of tightly and loosely self-trapped 2D solitons supported by the quasi-1D potential. The predictions (24) and (25) were also found to be in a reasonable agreement with numerical results. The formation of solitons in the 3D lattice potential is, generally, similar to the 2D case, although the relaxation time is shorter, due to the stronger interaction involving a larger number of adjacent cells. A typical example of a *stable* multi-cell 3D soliton, which forms itself in a strong lattice potential, is given in figure 9(a).

Additionally, in the full 2D lattice, stable solitons with *embedded vorticity* were found, despite the fact that the vorticity is not a dynamical invariant in the presence of the



**Figure 8.** Typical examples of stable single-humped (tightly bound, (a)) and multi-humped (loosely bound, (b)) two-dimensional solitons in the Gross–Pitaevskii equation with attraction, supported by the quasi-one-dimensional optical lattice.



**Figure 9.** (a) An established 3D soliton formed in a strong lattice potential. (b) An example of a two-dimensional vortex soliton with  $S = 1$ , supported by the square lattice. The vorticity is defined as  $S \equiv \Delta\phi/(2\pi)$ , where  $\Delta\phi$  is a change of the phase of the complex stationary wavefunction along a closed path surrounding the central point. The inset additionally shows the diagonal cross section of the vortex.

OL (see an example in figure 9(b)). In the 3D case, both the full 3D lattice and its quasi-2D counterpart support stable 3D solitons (Baizakov *et al* 2003, 2004a, 2004b), in contrast to the model with the ac-Feshbach-resonance modulation of the nonlinearity, which could not stabilize 3D solitons (Saito

and Ueda 2003, Abdullaev *et al* 2003). It is worth noting that the quasi-1D lattice cannot stabilize 3D solitons, or 2D vortex solitons (Baizakov *et al* 2004a, 2004b); however, a combination of the quasi-1D OL with the ac-Feshbach resonance gives rise to stable solitons in the 3D system (Matuszewski *et al* 2005).

A very recent result, reported by Kartashov *et al* (2004a, 2004b) is the existence of various stable solitons in the 2D medium equipped with a radial lattice, in the form corresponding to an OL that, as is well known (Durnin *et al* 1987), can be created by a diffraction-free Bessel beam. Moreover, the same 2D radial lattice may also support stable 3D solitons (Mihalache *et al* 2004c). The simplest version of the corresponding GP equation (8) in the 3D case is

$$i \frac{\partial u}{\partial t} = -\frac{1}{2} \left( \nabla_{\perp}^2 u + \frac{\partial^2 u}{\partial z^2} \right) - |u|^2 u - A J_0 \left( \frac{r}{r_0} \right) u,$$

where  $J_0$  is the Bessel function, with constant  $A$  and  $r_0$ , (actually, the potential  $\sim J_0$  in this equation assumes a mixture of a weak Bessel beam with a strong spatially uniform background (Mihalache *et al* 2004c)) and  $\nabla_{\perp}^2$  is the transverse Laplacian acting on the coordinates  $x$  and  $y$ , with  $r \equiv \sqrt{x^2 + y^2}$ .

The above BEC models, except for the one with the full 3D lattice (but including the radial Bessel lattice), also find direct interpretations in terms of nonlinear optics. Indeed, if  $t$  is realized as the propagation distance  $z$ , then the 3D model with the quasi-2D lattice applies to the propagation of light in a bulk Kerr (self-focusing) medium with the periodic transverse modulation of the refractive index in  $x$  and  $y$ . Thus, stable LBs can be predicted in such a setting. Indeed, stable one-parameter families of 3D spatiotemporal solitons in self-focusing cubic Kerr-type optical media with an imprinted 2D photonic lattice have been recently found by Mihalache *et al* (2004b). These 2D photonic lattices can support stable 3D solitons provided that their energy is within a certain interval and the strength of the lattice potential, which is proportional to the refractive index modulation depth, is above a certain threshold value (Mihalache *et al* 2004b). Remarkably, for sufficiently large values of the lattice strength parameter, the Hamiltonian–energy curves display *two cusps*, instead of a single one as in other 2D and 3D Hamiltonian systems (see, e.g., Akhmediev and Ankiewicz 1997). This unique feature is intimately related to the existence of *stable* 3D solitons within a finite interval of their energies. This two-cusp structure is actually an example of the so-called ‘swallow’s-tail’ structure, which is one of generic patterns known in the catastrophe theory. Actually, this pattern occurs quite rarely in physical applications (for a review on the catastrophe theory in the application to the soliton stability see, e.g., Kusmartsev (1989)).

Finally we stress that the 2D model with the 2D lattice pertains to spatial (rather than spatiotemporal) solitons in the same bulk medium. Further, the 2D model equipped with the quasi-1D lattice admits two different interpretations in terms of optics. If the coordinate  $y$  along the free (unmodulated) direction is replaced by the reduced time  $\tau$ , the latter model describes the  $(1+1+1)$ D spatiotemporal solitons in a planar waveguide with the refractive index periodically modulated in the transverse direction  $x$ . If, instead,  $y$  remains the second transverse coordinate, then one may consider the model as

describing  $(1+1+1)$ D spatial solitons in the bulk medium, where the refractive index is modulated only in one transverse direction,  $x$ .

#### 4.3. Spatiotemporal solitons in the model with the quadratic nonlinearity

**4.3.1. Spatiotemporal solitons versus collapse.** Long before the  $\chi^{(2)}$  solitons were observed experimentally, Karamzin and Sukhorukov (1976) had shown that stable 2D solitons exist in quadratic media. Later, Kanashov and Rubenchik (1981) proved, by using rigorous variational estimates, that in the 3D medium of this type, the *only* stable object may be a fully localized soliton, i.e., an STS of the  $(2+1+1)$ D type (their analysis did not consider the role of the phase mismatch). An implication of this result is that spatial  $(2+1)$ D cylindrical solitons are, strictly speaking, unstable too, against time-dependent modulational perturbations breaking their uniformity along the propagation axis; from this point of view, 1D and 2D solitons in infinitely long samples are only transient objects. As was explained above, the modulational instability of  $(1+1+1)$ D solitons in the bulk  $\chi^{(2)}$  was observed by Liu *et al* (2000b); see figure 4. The onset of an instability of this type was studied in numerical simulations of the  $\chi^{(2)}$  model by De Rossi *et al* (1997), and the modulational instability gain in a  $(1+1)$ D system (a planar waveguide) was experimentally measured by Schiek *et al* (2001).

**4.3.2. The variational approximation versus direct numerical results.** While multidimensional solitons in the  $\chi^{(2)}$  model should be looked for in a numerical form, useful quasi-analytical approximations can be applied too. In particular, an approximation based on a product ansatz for the soliton was proposed by Hayata and Koshihira (1993). A more accurate approach, based on the VA, was developed by Malomed *et al* (1997) (VA was earlier applied to STS in the model with a saturable nonlinearity by Enns and Rangnekar (1993)).

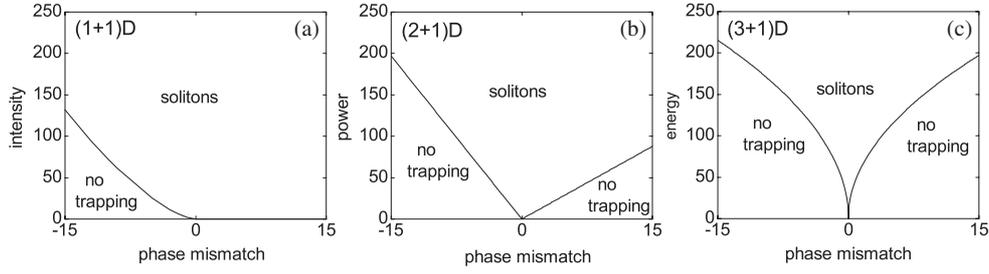
VA starts with the choice of an ansatz (trial function) for the stationary solutions  $U$  and  $V$ ; cf equations (19) and (23). As well as the above *ansätze*, a tractable one is based on the Gaussian form of the fields  $U$  and  $V$  in both spatial and temporal variables:

$$U = A \exp(-ar^2 - \rho\tau^2), \quad V = B \exp(-br^2 - \sigma\tau^2). \quad (26)$$

Here,  $A$ ,  $B$  and  $a$ ,  $\rho$ ,  $b$ ,  $\sigma$  are variational parameters, namely, amplitudes and inverse squared spatial and temporal widths of the FF and SH components of the STS, and  $r$  is the transverse coordinate in the  $(1+1+1)$ D case, or the transverse radial variable in the  $(2+1+1)$ D case.

Another ingredient of VA is the Lagrangian  $L$  corresponding to the stationary equations obtained from the dynamical system (10) and (11). In the case of  $c = 0$  and real  $U$  and  $V$ , and without the  $\chi^{(3)}$  terms ( $\Gamma = 0$ ), the Lagrangian is  $L = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} d\tau \mathcal{L}$  or  $L = \int_0^{\infty} r dr \int_{-\infty}^{+\infty} d\tau \mathcal{L}$  in the  $(1+1+1)$ D and  $(2+1+1)$ D cases, respectively, where the Lagrangian density is

$$\mathcal{L} = \frac{1}{2} (\nabla_{\perp} U)^2 + \frac{1}{8} (\nabla_{\perp} V)^2 + U_{\tau}^2 / 2 + \frac{\delta}{8} V_{\tau}^2 + \kappa U^2 + \left( \kappa + \frac{\beta}{2} \right) V^2 - U^2 V \equiv \mathcal{H} + \kappa \mathcal{E}, \quad (27)$$



**Figure 10.** Normalized threshold intensity (a), power (b), and energy (c) of the solitons in the second-harmonic-generation model versus the normalized phase-velocity mismatch.

$\mathcal{H}$  and  $\mathcal{E}$  being the corresponding Hamiltonian and energy densities.

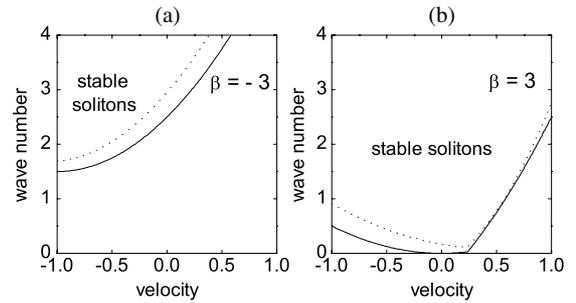
The Gaussian ansatz is inserted into the Lagrangian, and the integration is performed explicitly, which yields a *reduced Lagrangian* as a function of the variational parameters. Finally, the equations that these parameters obey are derived by equating to zero the derivatives of the effective Lagrangian with respect to them. As was mentioned above, the VA may also predict, in a part, stability of the solitons, using the VK criterion.

The VA has produced the following predictions for the (1 + 1 + 1)D and (2 + 1 + 1)D solitons in the  $\chi^{(2)}$  model:

- (i) For all values  $\delta > 0$  of the relative-dispersion parameter, there is exactly one stable soliton, in both the (1 + 1 + 1)D and (2 + 1 + 1)D cases.
- (ii) There is a finite minimum energy  $E_{\text{thr}}$  (*threshold*) necessary for the existence of the solitons.
- (iii) In the (1 + 1 + 1)D case, exactly one solution is predicted for  $\delta < 0$  too, i.e., for the case when the GVD is *normal* at SH.
- (iv) In the (2 + 1 + 1)D case, solutions for  $\delta < 0$  exist only in a very narrow stripe, e.g., for  $\delta > -0.034$  if  $\beta = -3/2$ . In this region, VA produces two different physical solutions, only one of which may be stable.

The above VA predictions are to be compared to results of direct simulations of the STS dynamics in the  $\chi^{(2)}$  model (Malomed *et al* 1997, Skryabin and Firth 1998, Mihalache *et al* 1998a, 1999a). The output of numerical calculations is illustrated in figure 10, where we plot the threshold characteristics for the one-, two-, and three-dimensional solitons ( $D = 1, 2, 3$ ), in the case of  $\delta = 1$ , as functions of the normalized phase-velocity mismatch  $\beta = k_1 \varrho_0^2 \Delta k$ , where  $\Delta k \equiv 2k_1 - k_2$ ,  $\varrho_0$  is the spatial width of the beam, and  $k_1$  and  $k_2$  are the FF and SH wavenumbers, respectively. A simple scaling analysis shows that, for positive and negative mismatches, the threshold behaves as  $|\beta|^{(4-D)/2}$ . It is observed that, for the positive mismatch, there is no threshold for the formation of (1 + 1)D solitons; moreover, the threshold vanishes at the exact phase-matching point ( $\beta = 0$ ) for any dimension  $D$ . Remarkably, for spatial solitons ( $D = 2$ ) and at the positive mismatch, an exact expression holds (Torner *et al* 1995b):  $P_{\text{thr}} = \beta P_{\text{NLS}}$ , where  $P_{\text{NLS}} \simeq 5.85$  is the known normalized collapse threshold of the 2D NLS equation (it is the same as the norm of the Townes soliton which was quoted above in a different normalization).

Comparison with numerical simulations shows that the above predictions (i) and (ii) are correct, aside from the fact



**Figure 11.** Regions of existence and stability of walking spatiotemporal solitons at negative (a) and positive (b) values of the phase mismatch.

that the numerically found threshold energy  $E_{\text{thr}}$  vanishes at the exact-matching point,  $\beta = 0$ ; see figure 10 (the latter circumstance is missed by VA).

The features (iii) and (iv), i.e., the existence or nonexistence of STS in the case when the dispersion is normal at SH, deserve special discussion. This issue is a physically significant one because, as was explained in detail above, in available  $\chi^{(2)}$  materials it is quite difficult to find a case when GVD is anomalous at both FF and SH simultaneously, without strong dissipation.

Initial calculations performed by Mihalache *et al* (1998a) did not produce STS in the case when the GVD at SH was normal. However, semi-analytical and numerical studies reported by Towers *et al* (2003) suggest that there is a well-defined region in the parameter plane ( $\beta, \delta$ ), where stable spatiotemporal soliton-like pulses, of both (1 + 1 + 1)D and (2 + 1 + 1)D types, exist. In the rigorous sense, those pulses may not be solitons, as their SH component may have an attached ‘tail’, which does not vanish at infinity. Nevertheless, numerical computations show that, for small enough normal GVD, the tail’s amplitude, if any, is smaller than the peak amplitude of the pulse by a factor of (at least)  $\sim 10^4$ , and hence the pulses may be identified as solitons in the experiment. A related issue, which was investigated by Beckwitt *et al* (2003), is the possibility of the existence of stable temporal (1D) solitons in the same case, with the result that an effective existence border is found at still larger values of the normal GVD at SH. Notice that the existence of long-lived soliton-like objects at small and even at moderate normal GVD at the SH, with anomalous GVD at the FF, was also predicted by Torner (1999). These results indicate the possibility of experimental search for STS-like objects under such conditions.

The effects of the GVM on STS were considered in detail by Mihalache *et al* (1999b) for the  $(1 + 1 + 1)$ D case, and by Mihalache *et al* (2000c) for the  $(2 + 1 + 1)$ D case. One-parameter and two-parameter families of *walking* STS were found to exist, most of them being stable. These families of solitons are natural generalizations of 1D and 2D walking solitons introduced by Torner *et al* (1996), Etrich *et al* (1997), and Mihalache *et al* (1997). As in the spatial case, it was found that, in most cases, the stationary STS with the temporal walk-off continue to exist and remain stable, if they were stable at zero GVD ( $c = 0$ ). This feature is illustrated by figure 11 for  $c = 1$  and  $\delta = 0.5$ , which demonstrates that most of the two-parameter families of the 3D walking solitons, characterized by the corresponding soliton velocity and nonlinear wavenumber shift, are stable. The walking solitons are unstable only in a narrow strip between the dotted lines in figure 11 and the solid lines which delineate their existence border (notice the asymmetry of the stability domains with respect to the sign of the phase mismatch  $\beta$ ).

The effect of the temporal GVM is more conspicuous in terms of the threshold energy  $E_{\text{thr}}$ , necessary for the existence of the STS:  $E_{\text{thr}}$  no longer vanishes at the zero-phase-mismatch point ( $\beta = 0$ ) if  $c \neq 0$ . In the absence of GVM, simple scaling arguments predict that  $E_{\text{thr}}$  must be proportional to  $|\beta|^{1/2}$  for both  $\beta > 0$  and  $\beta < 0$ . In the case of  $\delta < 0$ , a maximum value of GVM was found by Towers *et al* (2003), up to which the soliton-like STS, found for normal GVD at SH, continues to exist.

To summarize, whole families of stable STS exist above the threshold value of the light intensity, and almost all members of the soliton families are stable. However, this does not guarantee that it is easy to generate the solitons; in fact, suitable conditions for their generation still must be found. For example, for a given material and input light conditions, increase of the input intensity does not necessarily improve efficiency of the soliton generation (Torner *et al* 1999, Carrasco *et al* 2002). In any case, the key step for the generation of the LBs is identification of suitable materials, pump wavelengths and pulse conditions, for which all the necessary conditions, as concerns GVD, GVM, losses, etc, are fulfilled.

#### 4.4. Spinning solitons

A distinct species of multidimensional solitons is the one with intrinsic vorticity (often referred to as ‘spin’). Unlike the above-mentioned vortex solitons on square lattices, in isotropic media the spin is related to the angular momentum, which is a dynamical invariant. Both  $(2 + 1)$ D spatial solitons in an isotropic bulk medium and  $(2 + 1 + 1)$ D STS may carry the spin. Finding a stationary shape of spinning solitons is not difficult, but their stability is an issue. Spinning spatial solitons in models containing a single type of the nonlinearity, such as quadratic or saturable, are strongly unstable against azimuthal perturbations breaking the axial symmetry (Kruglov *et al* 1992, Torner and Petrov 1997, Firth and Skryabin 1997, Torres *et al* 1998). As a result, they split into sets of stable zero-spin solitons, which move away, so that the initial spin momentum of the unstable soliton is converted into the orbital momentum of the splinters. The spontaneous instability of the spinning  $(2+1)$ D spatial solitons was observed experimentally

in both saturable media (Tikhonenko *et al* 1995, Bigelow *et al* 2004) and in quadratic nonlinear media (Petrov *et al* 1998). When the instability is externally induced, it can be used to implement the concept of *soliton algebra* (Minardi *et al* 2001). In these experiments, vorticity was impressed on the input beams by means of holographic phase masks. Similarly, a spinning  $(2 + 1 + 1)$ D STS in the  $\chi^{(2)}$  model was numerically found to be unstable against azimuthal perturbations, that also splits it into a set of separating zero-spin solitons (Mihalache *et al* 2000b).

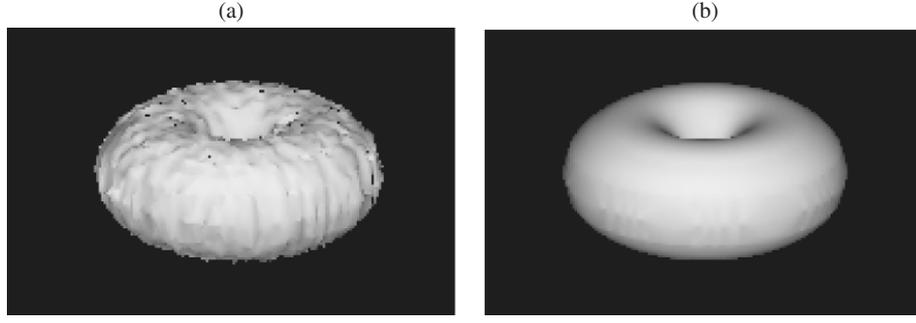
As was inferred, in a general form, by Mihalache *et al* (2002a), spinning STS may be stabilized if the model incorporates *competing* (self-focusing and self-defocusing) nonlinearities. Examples are the above-mentioned CQ model, see equation (7), and the quadratic–cubic one, based on equations (10) and (11) with  $\Gamma > 0$ . In this context, it is relevant to mention that the strength of the effective cubic nonlinearity in  $\chi^{(2)}$  crystals may be tuned by means of optical rectification (generation of a nearly dc field component, as shown by Torres *et al* (2002a, 2002b)).

For the first time, the existence of stable spinning spatial  $(2 + 1)$ D solitons with spin  $S = 1$  in the CQ model was demonstrated by means of direct simulations by Quiroga-Teixeiro and Michinel (1997). Later, the issue was studied in detail by means of accurate numerical computations of linear-stability eigenvalues for the corresponding stationary soliton solutions (Towers *et al* 2001b, Malomed *et al* 2002). The existence of stable spinning spatial solitons (also referred to as ‘ring vortices’) in the quadratic–cubic medium was shown numerically, within the framework of the same approach, by Towers *et al* (2001a). In both models, stable spinning solitons share common features: they have relatively large stability regions for  $S = 1$  and 2, and much narrower ones for  $S \geq 3$  (Pego and Warchall 2002, Mihalache *et al* 2004d, 2004e, Davydova and Yakimenko 2004).

In the latter context, it is relevant to mention that, in the 2D discrete NLS equation, stable vortex soliton may have  $S = 1$  (Malomed and Kevrekidis 2001) and  $S = 3$  (Kevrekidis *et al* 2004a), while the discrete solitons with  $S = 2$  are unstable (but stable quadrupole solitons can be found in lieu of them). Additionally,  $S = 1$  vortices with a phase shift  $\pi$  between them (i.e., opposite signs of the wave fields) can form stable bound states on the discrete lattice (Kevrekidis *et al* 2001).

The stability of the spinning  $(2+1+1)$ D solitons in the CQ model was investigated in detail by Mihalache *et al* (2002a) (the stationary shape of these solitary waves can be found not only in a numerical form (Mihalache *et al* 2000a), but also in an approximate semi-analytical one by means of VA, which is based on the ansatz  $U(r, \tau) = V(r) \text{sech}(\mu\tau)$ , as was shown by Desyatnikov *et al* (2000)). The soliton has a toroidal (‘doughnut-like’) shape; see figure 12. The one-parameter family of STS can be characterized by the dependence between the wavenumber  $\kappa$  and energy; see equations (16) and (12). Noteworthy features of STS in CQ nonlinear media can be summarized as follows:

- (i) There is an energy threshold necessary for their existence, which steeply increases with the spin (see figure 5).



**Figure 12.** Recovery of a stable spatiotemporal soliton with  $S = 1$  propagating in media with competing nonlinearities, which was strongly perturbed at input. (a) Isosurface plots of the initial perturbed soliton; (b) the self-cleaned output soliton.

- (ii) In the case of  $S = 0$ , the VK criterion is sufficient for the stability. However, for the spinning solitons it does not guarantee stability against perturbations destroying the axial symmetry of the ‘doughnut’; see further details below.
- (iii) The solutions exist in a finite interval of the wavenumbers,  $0 < \kappa < \kappa_{\max} = 3/16$ , the limit of  $\kappa \rightarrow \kappa_{\max}$  corresponding to  $E \rightarrow \infty$  (see figure 5). The divergence of the energy in this limit takes place via swelling of the outer radius of the ‘doughnut’, while the amplitude of the field remains limited. At the point  $\kappa = \kappa_{\max}$  the STS goes over into an infinitely extended background field with embedded vorticity and a hole in the centre, i.e., a *dark* vortex soliton.

As was stated above, stability is the central issue in the study of the spinning solitons. Qualitative information about the stability is provided by numerical propagation of solutions in the presence of random perturbations, and quantitative information is provided by numerical computation of the corresponding eigenvalues for small perturbations obeying the equations linearized about the stationary solution. In the CQ model, a general solution for the perturbation eigenmodes can be sought for in a self-consistent form,

$$\begin{aligned}
 u(Z, r, T, \theta) - U(r, \tau) \exp[i(S\theta + \kappa z)] \\
 = f_n(r, \tau) \exp\{\lambda_n z + i[(S+n)\theta + \kappa z]\} \\
 + g_n^*(r, \tau) \exp\{\lambda_n^* z + i[(S-n)\theta + \kappa z]\}, \quad (28)
 \end{aligned}$$

where  $n > 0$  is an integer azimuthal index of the eigenmode  $(f_n, g_n)$ , and  $\lambda_n$  is its instability growth rate. The self-consistency of the expression (28) means that the two azimuthal harmonics  $\exp[i(S+n)\theta]$  and  $\exp[i(S-n)\theta]$  form a closed system. Numerical calculations show that the most persistent unstable eigenmode has  $n = 2$ , for both  $S = 1$  and 2. With the increase of  $\kappa$ , the  $S = 1$  soliton becomes stable at  $\kappa = \kappa_{\text{st}} \approx 0.13$ , and the stability region extends up to  $\kappa_{\max} = 3/16$ , which limits the existence region of the STS. In other words, stable spinning solitons must have a large size and large energy.

Direct simulations completely corroborate the stability of the  $(2+1+1)$ D solitons with  $S = 0$  and 1 in the CQ model in the region where they are predicted to be stable. Moreover, a stable STS with  $S = 1$  can easily self-trap from initial configurations of a rather arbitrary shape (in particular, from the one containing a large random-noise component) with embedded vorticity, provided that the energy is large

enough. In figure 12 we show a typical self-cleaning process of the initially strongly perturbed stable soliton with  $S = 1$ . The existence of stable spinning STS in optical media with competing nonlinearities is a generic feature: stable 3D spinning solitons with vorticity  $S = 1$  supported by competing quadratic and cubic nonlinearities were found to exist too (Mihalache *et al* 2002b). Stable 2D and 3D two-component solitons of the same type (with  $S = 1$ ) were also found in a bimodal CQ system (Mihalache *et al* 2002c, 2003b).

It is relevant to mention that dissipative systems, described by the complex CQ Ginzburg–Landau equation, support stable 2D spinning solitons too (Crasovan *et al* 2001a, 2001b). However, as is always the case with solitons in dissipative models, these ones do not form a continuous family. Instead, for given parameters of the equation, just two soliton solutions exist, one stable and one unstable.

The conservative model with the CQ nonlinearity was recently also demonstrated to support extremely stable patterns in the form of circular *clusters* of solitons (also called ‘necklaces’; see also below). The necklaces can carry the angular momentum (Mihalache *et al* 2003a, 2004a). However, it remains unknown whether a narrow stability region exists for spinning STS with higher values of the vorticity,  $S > 1$ .

A recent experimental advance towards the observation of stable spinning STS is the stable propagation of spatially localized optical vortices with topological charge  $S = 1$  and  $S = 2$  in a self-focusing photorefractive strontium barium niobate (SBN) crystal (Neshev *et al* 2004, Fleischer *et al* 2004). These 2D vortices are created by self-trapping of partially incoherent light carrying a phase dislocation, and they can be stabilized when the spatial incoherence of light exceeds a certain threshold value (Jeng *et al* 2004). However, these vortices were created on top of a virtual lattice induced by a square-lattice grid of transverse laser beams illuminating the sample. Thus, the system is far from being isotropic, and may be compared to the GP equation with the square OL considered above. Stable necklace-shaped soliton clusters in the photorefractive medium equipped with the photonic lattice were also observed (and modelled theoretically) very recently by Yang *et al* (2005).

To conclude the consideration of spinning solitons, we notice that the prediction of the existence of stable 3D spinning solitons with a large size, and, accordingly, large energy in optical media with competing nonlinearities is, as a matter of fact, the second example (after *Skyrmions*, providing for

a classical-field model of nucleons) of spinning 3D localized objects known in physics.

#### 4.5. Interactions between spatiotemporal solitons

Collisions and other forms of interaction between STS are an important issue, but little has been thus far done on this topic (Stegeman and Segev 1999). However, collisions between  $(2 + 1 + 1)$ D LBs in the model with the saturable nonlinearity were simulated by Edmundson and Enns (1993), who had found that the collisions seem completely elastic, with the solitons reappearing virtually unscathed after the collision.

An analytical approach to the interaction problem is possible if the solitons interact at a large distance, so that overlapping between an exponentially decaying ‘tail’ of one soliton and the ‘body’ of the other may be treated as a small perturbation. As was shown by Malomed (1998), an effective potential of the interaction between well-separated identical zero-spin solitons with a phase difference  $\psi$ , temporal separation  $T$ , and the distance between their centres  $R$ , is

$$W(\Xi, \psi) = \text{const} \cdot \Xi^{-(D-1)/2} \exp(-\sqrt{2\kappa}\Xi) \cos \psi, \quad (29)$$

where  $\Xi^2 \equiv R^2 + T^2$ , and  $\kappa$  is the same wavenumber as in equation (16);  $D = 3$  and  $2$  correspond, as before, to the  $(2 + 1 + 1)$  and  $(1 + 1 + 1)$  cases, respectively. Using the potential (29), one can derive equations of motion for  $\Xi$  and  $\psi$ . An important peculiarity of the system is that the effective mass corresponding to the phase degree of freedom,  $\psi$ , is *negative* (hence local minima of the interaction potential are saddle points, rather than stable equilibria); see further details in the review by Malomed (2002). In a similar way, the interaction potential was derived by Maimistov *et al* (1999) for solitons belonging to different components in a bimodal system based on nonlinearly coupled CQ equations; in that case, the potential does not depend on the phase difference between the solitons.

## 5. Conclusion and open problems

Thus far, spatiotemporal solitons were investigated theoretically only in basic models. Many challenging possibilities were not yet explored, such as spinning solitons whose spin is *not* aligned with the propagation direction, or anisotropic spinning solitons containing vortices with an intrinsic spatial structure, as suggested by Freund (1999) and Molina-Terriza *et al* (2001). Another interesting possibility is to search for spinning  $(2 + 1 + 1)$ D spatiotemporal solitons in the model of a  $\chi^{(2)}$  medium equipped with the Bragg grating (the model was introduced by He and Drummond (1998)). A somewhat simpler but also exciting problem is the question of stability of composite solitons with opposite spins ( $S, -S$ ) in bimodal systems. A very recent result by Desyatnikov *et al* (2005) is that such solitons with *hidden vorticity* may be stable in certain domains of their existence region. A similar problem was also considered by Leblond *et al* (2005) for the  $\chi^{(2)} : \chi^{(3)}$  model, with a conclusion that hidden-vorticity solitons may be, at least, very long lived in this case too.

Given experimental challenges in the creation of an environment that may support STS, a possibility that LBs might emerge from a dynamical instability becomes attractive. In this regard, it is relevant to mention an ongoing study

of the so-called nonlinear *X-waves* (Di Trapani *et al* 2003). This is a generalization of the diffraction-free propagation of Bessel beams in a linear medium (Durin *et al* 1987) to the polychromatic case (Lu and Greenleaf 1992, Sonajalg *et al* 1997). Linear wavepackets of the form

$$\exp(iS\theta) \int_0^\infty B(k) J_n(kr \sin \zeta) \exp[ik(z \cos \zeta - ct)] dk, \quad (30)$$

where  $(r, \theta, z)$  are cylindrical coordinates,  $S$  is an integer,  $B(k)$  is any smooth function, and  $J_n$  is the usual Bessel function, undergo spatiotemporally invariant propagation (Lu and Greenleaf 1992). Experimentally, such ‘X-waves’ are launched in the form of a conical radiation pattern with the cone angle  $\zeta$ . They propagate linearly without spreading in time or space, but, off-axis, their envelopes are only weakly localized, in contrast to the exponentially localized soliton solutions; therefore they contain infinite energy. In the experiment, the main difficulty is production of the necessary conical radiation pattern.

Di Trapani *et al* (2003) discovered that a quasi-localized light pattern can spontaneously emerge from unstable propagation of a short pulse in a  $\chi^{(2)}$  crystal with normal GVD and large group- and phase-velocity mismatches. A pulse with the temporal and transverse sizes  $\sim 200$  fs and  $\sim 1$  mm produced a stably propagating pattern with the size of  $25$  fs  $\times$   $25$   $\mu$ m, which kept a few per cent of the input energy. This pulse exhibits characteristics of an X-wave (Trull *et al* 2004). Trillo *et al* (2002) have shown that X-waves are indeed solutions of the coupled wave equations with normal GVD and large phase mismatch. The  $\chi^{(2)}$  nonlinearity may be helpful in this context: phase-mismatched SH generation naturally produces conical emission, and is thus an ideal way to launch the X-waves.

Although the nonlinear X-waves are not localized solutions and thus not LBs in the usual sense, they do exhibit a kind of 3D spatiotemporal self-focusing and trapping. Studies of X-waves will complement work aimed at the production of true STS, as nonlinear X-waves can form in systems where solitons cannot, e.g., when GVM is large and GVD is normal. A challenging issue that remains unexplored is a continuous transition between true STS and X-waves. In the latter connection, it is relevant to mention that the X-wave solutions for the case of the *anomalous* temporal dispersion, as well as for two-dimensional X-waves (that were not considered before), were recently reported by Christodoulides *et al* (2004).

In the last few years a lot of attention was devoted to transverse dynamics in nonlinear optics in cavities, namely, the study of transverse effects in passive and active optical cavities, such as the pattern formation and localized structures in dissipative systems (for a recent overview see Mandel and Tlidi (2004)). In such systems the prototype dynamical equations are the complex Ginzburg–Landau and Swift–Hohenberg equations (Aranson and Kramer 2002). The competition between transverse diffraction and nonlinearities in optical resonators leads to the formation of cavity solitons appearing as bright spots on a homogeneous background. Recently the possibility of using these cavity solitons as self-organized micropixels in semiconductor microresonators has been demonstrated (Hachair *et al* 2004). Moreover, the formation of 3D localized structures in nonlinear optical

resonators, confined in both the transverse plane and in the longitudinal direction (the so-called cavity LBs), beyond the usual mean-field limit, has been recently predicted by Brambilla *et al* (2004).

An important open frontier of the field is the study of LB complexes, built from several stable solitons. Structures investigated to date, such as soliton necklaces introduced by Soljačić *et al* (1998), Soljačić and Segev (2001), or soliton clusters studied by Desyatnikov and Kivshar (2002), tend to expand or self-destroy in the presence of small perturbations. However, recent numerical investigations indicate that competing quadratic–cubic nonlinearities allow one to build more robust, yet still metastable structures (Kartashov *et al* 2002, Crasovan *et al* 2003a). As was mentioned above, the use of the CQ nonlinearity gives rise to very robust spatiotemporal necklaces, as shown by Mihalache *et al* (2004a). Completely stable stationary necklace clusters of spatial solitons, supported by photonic lattices in photorefractive media, were recently created experimentally and explained theoretically by Yang *et al* (2005).

Soliton formation in discrete and quasi-discrete settings has been addressed only briefly here (for a recent review of the main features afforded by waveguide arrays, see Christodoulides *et al* (2003)). However, this is a subject of intense theoretical and experimental research (see, e.g., Eisenberg *et al* 1998, Efremidis *et al* 2002, Fleischer *et al* 2003, Fleischer *et al* 2004, Martin *et al* 2004, Neshev *et al* 2004, Yang and Musslimani 2003, Musslimani and Yang 2004, Iwanow *et al* 2004, Xu *et al* 2004) motivated by potential applications of discrete solitons to light manipulation and control, and by new opportunities suggested by the recent demonstration of the optically induced photonic lattices mentioned above. The study of spatiotemporal solitons in such lattices gives rise to important questions open for future research.

The results obtained in optics may offer clues in the search for multidimensional solitons in other areas of physics. As was said above, the field which is closest to nonlinear optics in terms of the theoretical description, and which has attracted a great deal of attention in the last few years, is the Bose–Einstein condensation (BEC). While only 1D solitons, bright, dark, and, very recently, ones of the gap type (Eiermann *et al* 2004), have thus far been observed in BEC (for the ordinary dark and bright solitons in BEC, see reviews by Anglin and Ketterle (2002), and by Anderson and Meystre (2002)), the recently reported theoretical results, briefly reviewed in this article, suggest possibilities for the creation of stable 2D and 3D solitons in the attractive BEC by means of optical lattices. Moreover, the use of the lattices opens a possibility to create multi-dimensional gap solitons even in repulsive BEC (Baizakov *et al* 2002). In connection with this, the existence of spatially localized topological states of a BEC with repulsive atomic interactions confined by an optical lattice (the so-called matter-wave gap vortices) has been predicted by Baizakov *et al* (2004a), Sakaguchi and Malomed (2004) and Ostrovskaya and Kivshar (2004). Recently, a recipe for constructing globally linked three-dimensional structures of topological defects hosted in trapped wave fields, such as vortex dipoles, vortex quadrupoles, and parallel vortex rings, has been given (Crasovan *et al* 2002, 2003c, 2004). These vortex clusters exist as robust stationary, highly excited

collective states of nonrotating BECs and are dynamically and structurally stable. It is also relevant to mention that stable ‘supervortices’, i.e., ring arrays of fundamental ( $S = 1$ ) vortices, with global vorticity imprinted onto the entire array, were recently predicted by Sakaguchi and Malomed (2005) in the 2D Gross–Pitaevskii equation with the attractive nonlinearity and square lattice.

Localized X-matter waves in three dimensions formed by an ultracold Bose gas in an optical lattice, characterized by a peculiar biconical shape, have been predicted too (Conti and Trillo 2004). It is also interesting to mention a formal analogy between the 3D optical model with the  $\chi^{(2)}$  nonlinearity and models of mixed atomic–molecular BEC. The analogy suggests a principal possibility of the existence of *self-supporting* soliton-like condensates in coherent mixtures of atoms and molecules (Julienne *et al* 1998, Drummond *et al* 1998, Javanainen and Mackie 1998, Cusack *et al* 2001).

To conclude, after many years of theoretical progress, observation of truly  $(2 + 1 + 1)$ D solitons, both spinless and spinning, remains a great challenge. Physical settings described by suitable mathematical models need to be identified in terms of existing materials. Spin-offs of the research motivated by the optical STSs have important applications and implications for other areas of physics.

## Acknowledgments

We appreciate valuable collaborations and discussions with many colleagues around the world. The list is too long to be included here. The work which led to this review was supported, in part, by the Binational (US–Israel) Science Foundation under grant No 1999459. BAM appreciates a partial support from the Israel Science Foundation through grant No 8006/03. The work of DM was supported in part by Institució Catalana de Recerca i Estudis Avançats (ICREA), Barcelona, Spain. FW acknowledges the support of the US National Science Foundation under award PHY-0099564. LT acknowledges support by the Spanish Government under contract BFM2002-02861.

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