The spatial shape of entangled photon states generated in non-collinear, walking parametric downconversion

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Abstract
We address the spatial shape of entangled photon states generated in spontaneous parametric downconversion in non-collinear geometries, when the interacting waves exhibit Poynting vector walk-off. We discuss how such shape depends on the interplay between the state ellipticity caused by the non-collinear geometry, and the Poynting vector walk-off. In particular, we study the impact of the new features exhibited by walking entangled photons on the implementation of quantum protocols based on the spiral spectra of the entangled photons.

Keywords: parametric downconversion, orbital angular momentum of light

Spontaneous parametric downconversion (SPDC) is a reliable source of entangled photons. The generated two-photon states exhibit, in particular, spatial entanglement embedded in the corresponding mode function. The spatial structure of the two-photon state provides a new resource to explore quantum physics beyond the two-dimensional Hilbert space generated by the polarization state of the photons. The corresponding multidimensional entangled states, or qudits, can be encoded in orbital angular momentum (OAM) [1], thus providing infinite-dimensional alphabets [2], which can be used to conduct proof-of-principle demonstrations of quantum protocols whose implementation requires higher-dimensional Hilbert spaces. Illustrative examples include the violation of Bell inequalities with qutrits [3], and the implementation of a quantum coin tossing protocol [4].

Two-photon entangled states with well defined spatial properties, e.g., qutrits encoded in a well defined value of OAM [5, 6], are not harvested directly at the output of the downconverting crystal, thus elucidation of techniques to generate the appropriate states are required. In general, the implementation of a \(d\)-dimensional quantum channel requires the generation of arbitrary engineered entangled states, thus controlling such structures is of paramount importance for many applications, including, e.g., those addressed to multidimensional quantum imaging [7]. An appropriate tailoring of the mode function of the two-photon state requires the consideration of all parameters that effectively determine its spatial shape. Importantly, in certain regions of the parameter space that define the SPDC process, some of these parameters might be neglected, or alternatively have their importance enhanced.

In a typical SPDC configuration, a strong pump interacts with a nonlinear \(\chi^{(2)}\) crystal, generating a pair of photons that propagate at a certain angle. Most previous investigations of entangled photons addressed nearly collinear phase-matching geometries in which pump, signal and idler photons propagate almost along the same direction. Actually, in genuine non-collinear geometries both the spin angular momentum [8], and the orbital angular momentum of the entangled photons depend strongly on the propagation direction of the photons. The impact of such effects on the OAM can be made important even when only purely geometrical features are considered [9], and can become dominant in the case of highly non-collinear settings such as transverse-emitting configurations [10], or for highly focused pump beams [11]. The presence of Poynting vector walk-off due to crystal birefringence is also expected to greatly impact the spatial structure of the entangled quantum state, as we confirm here.

Here we show that the spatial shape of the two-photon state results from the interplay between the effects on the spatial shape due to the non-collinear geometry and due to the Poynting vector walk-off. The importance of both effects
The electric field amplitude operator corresponding to the signal photon $E_s$ can be written as $E_s^\dagger(x, z, \tau) \propto \int dp \exp(-ik_z z - ip \cdot x) a_s^\dagger(p)$, where $x_1 = (x_1, y_1, z_1)$ is the position in the transverse plane of the signal photon, and $a_s^\dagger$ is the creation operator of a signal photon with transverse wavevector $p$. For the idler photon, we can write a similar expression, with $x_2 = (x_2, y_2)$ being the transverse coordinates, $q_1 = (q_1, q_2)$ the transverse wavevector, $k_i = \sqrt{(k_0^q)^2 - |q|^2}$ the longitudinal wavenumber, and $k_0^q = \omega_{ni}(\nu_0/c)$. The quantum state of the two-photon state (excluding vacuum) is written [10, 13]

$$|\Psi\rangle = \int dp \, dq \, \Phi(p, q) a_s^\dagger(p)a_i^\dagger(q)|0, 0\rangle,$$

with

$$\Phi(p, q) = E_0(p, s + q_i, \Delta_0) \text{sinc}(\Delta_L/2) \exp(i\Delta_L/2).$$

where $L$ is the crystal length, $\Delta_0 = k_0^q - k_i \cos \phi_1 - k_i \cos \phi_2 + (p_s + q_i) \tan \rho_0 \cos \alpha + \Delta_0 \tan \rho_0 \sin \alpha - p_s \sin \phi_1 - q_i \sin \phi_2$, comes from the phase matching condition in the $z$ direction, and

$$\Delta_0 = p_s \cos \phi_1 + q_i \cos \phi_2 - k_i \sin \phi_1 - k_i \sin \phi_2.$$
For a collinear configuration ($\varphi_1 = \varphi_2 = 0$), and neglecting the Poynting vector walk-off of all interacting beams, the mode function given in equation (2) is written [15, 16] $\Phi(p, q) \propto E_0(p + q) \text{sinc}(|p - q|^2 L/(4k_0^2)) \exp[-|p|^2 L/(4k_0^2)]$, where one should make use of the paraxial approximation $k_0 \approx |p|^2/(2k_0^2)$, and correspondingly for the signal and idler wavevectors. For this case, it has been shown [16] that if the spatial shape of the pump beam corresponds to a vortex beam with winding number $l_0$, and we project the idler photon into a mode with winding number $l_2$, the OAM content of the signal photon present a single peak at $l_1$, so that it fulfills the selection rule

$$l_0 = l_1 + l_2.$$  

A similar result has been obtained under the thin crystal approximation [17], where the mode function is now written as $\Phi(p, q) \propto E_0(p + q) \text{sinc}(|p - q|^2 L/(4k_0^2)) \exp[-|p|^2 L/(4k_0^2)]$. The signal and idler photons traverse two identical 2-f systems, with focal length $f_2 = f_1 = f$. The observation planes are assumed to be located at the focal length of the lens, $z_1 = z_2 = f$. The 2-f system provides a spatial image of the two-photon state $\Phi(p, q)$, so that by measuring coincidence rates at different positions in the plane one obtains information about the spatial shape of the mode function [18]. The probability to detect an idler photon at position $x_2$ in coincidence with a signal photon at position $x_1$ is given by $R_c(x_1, x_2) = |\Phi(2\pi x_1/\lambda f, 2\pi x_2/\lambda f)|^2$. If the idler photon is projected into a plane-wave mode with $q = 0$, which is experimentally realized by locating a small pinhole at $x_2 = 0$ in the focal plane, the mode function $\Phi_s(p)$ of the signal photon turns out to be $\Phi_s(p) = \Phi(p, q = 0)$. Projection into the mode $q = 0$ is equivalent to projection of the idler photon into a Gaussian mode with a very large beam width, i.e., $\Phi_s(p) \propto \int dq \Phi(p, q) \exp(-|q|^2 u_1^2/4)$, where $u_1 \to \infty$ is the width of the Gaussian mode. Thus the configuration described here performs a projection of the idler photon onto a mode with $l_2 = 0$.

In figures 2 and 3 we plot the spatial shape of the signal photon. More specifically, we plot the coincidence rate $R_c(x_1, x_2 = 0)$, which gives us the intensity of the mode function. As an illustrative example, we consider a LiIO 3 crystal with length $L = 2$ mm. The pump beam is written $E_0(p) \propto (p_x + ip_y) \exp(-|p|^2 w_0^2/4)$, which corresponds to a vortex beam with winding number $l_0 = 1$ and pump beam width $w_0$. In all cases considered, the diffraction length of the pump beam, $L_d = k_0^2 w_0^2/2$, is assumed to be much larger than the crystal length ($L_d \gg L$). Under this condition, if we make use of the paraxial approximation in equations (3) and (4), the mode function of the signal photon can be written as

$$\Phi_s(p) \propto E_0(p_x, p_y \cos \varphi_1) \text{sinc} \left\{ \left[ p_x \tan \rho_0 \cos \alpha \right. \right.$$  

$$\left. + p_y \left( \tan \rho_0 \cos \varphi_1 \sin \alpha - \sin \varphi_1 \right) \right\}^{L/2} \right.$$  

$$\times \exp \left\{ i \left[ p_x \tan \rho_0 \cos \alpha \right.$$  

$$\left. + p_y \left( \tan \rho_0 \cos \varphi_1 \sin \alpha - \sin \varphi_1 \right) \right\}^{L/2}. \right\}$$  

(6)

Therefore, the intensity of the mode function of the signal photon, $|\Phi_s(p)|^2$, is strongly affected by the slope of the ($p_x, p_y$) plane of the loci of perfect phase matching transverse momentum, which is written

$$\frac{p_x}{p_y} = \frac{\sin \alpha - \tan \rho_0 \cos \varphi_1 \sin \alpha}{\tan \rho_0 \cos \alpha}.$$  

(7)

As is readily apparent, this expression turns out to be determined by the relationship between three angles: the angles that determine the direction of emission of the downconverted photons, $\varphi_1$ and $\alpha$, and the walk-off angle of the pump beam, $\rho_0$.

In figure 2, we plot the spatial shape of the signal photon for some representative cases corresponding to $\alpha = 0^\circ$. In this case, the downconverted photons propagate in a plane perpendicular to the plane formed by the optics axis and the direction of propagation of the pump beam. Perfect phase matching is achieved for values of the transverse momentum so that $p_x/p_y = L_w/L_{nc}$, where $L_{nc} = w_0/\sin \varphi_1$ is the non-collinear length and $L_w = w_0/\tan \rho_0$ is the walk-off length of the pump beam. In figure 2(a), the width of the pump beam is very large ($w_0 = 800 \mu m$), so that both the non-collinear length and the walk-off length are much larger than the crystal length ($L \ll L_{nc}$, $L_w$). The coincidence rate
phase matching corresponds to methods [23]. The mode function given in equation (1) is holographic and filtering techniques [1, 22], or interferometric modes can be resolved experimentally using combinations of operator [21]. The topological phase structure of these modes is naturally expressed in terms of eigenstates of the paraxial OAM operator, namely, the OAM content of the mode function that describes the signal photon. The OAM content [2] is given by the distribution \( C_n = \int_{0}^{\infty} \rho_k \, d\rho_k \, |a_n(\rho_k)|^2 \),

\[
a_n(\rho_k) = 1/(2\pi)^{1/2} \int_{0}^{2\pi} d\varphi_k \, \Phi_l(\rho_k, \varphi_k) \exp(-i\varphi_k),
\]

with \( \rho_k = |p| \) and \( \varphi_k = \tan^{-1} p_x/p_y \) being the radius and azimuthal phase in cylindrical coordinates, respectively, of the transverse wavevector \( p \).

In figure 4 we plot the OAM content of some SPDC configurations with crystal length \( L = 2 \) mm and \( l_0 = 1 \). Figures 4(a)–(c) correspond to an angle of emission of \( \varphi_1 = 1^\circ \) and \( \alpha = 0^\circ \). For a beam width of \( w_0 = 800 \mu m \), which is plotted in figure 4(a) and whose spatial shape is plotted in figure 2(a), the OAM content shows a distribution highly peaked at \( l_1 = 1 \). The OAM content of the downconverted photons can change dramatically when modifying the beam width of the pump beam, as shown in figures 4(b) and (c). The OAM distribution plotted in figure 4(c) corresponds to the spatial shape plotted in figure 2(c). Only for large values of \( w_\text{lc} \) and \( L_\text{lc} \), when compared with the crystal length \( L \), the OAM distribution of the signal photon shows mostly a single peak, according to the selection rule \( l_0 = l_1 + l_2 \), as is the case in some experiments [1, 12]. In some other experiments, the parameter space of the experimental configuration chosen determines a smaller value of \( L_\text{lc} \), thus observing a non-centrosymmetric spatial shape of the signal photon [11, 13]. Therefore, the OAM distribution of the mode function of the signal photon contains many more modes, for which \( l_0 \neq l_1 + l_2 \). In figure 4(d) is shown the OAM content corresponding to the spatial shape plotted in figure 3(b), with \( \alpha = 90^\circ \) and \( \varphi_1 = 5^\circ \). Notice that although one has \( w_\text{lc} < L_\text{lc} \), the OAM distribution shows a single peak at \( l_1 = 1 \), due to the compensation of the non-collinear and spatial walk-off effects. The important practical implications of this result are readily apparent.

The main conclusion to be drawn from the previous results is that when implementing spatially encoded quantum information protocols based in OAM with strongly focused pump beams, or alternatively longer crystals, the specific properties afforded by walking entangled states cannot be avoided. As shown in figure 4, such properties are not necessarily detrimental. Notice also that strongly focused beams or longer crystals can be required to obtain a higher count rate of downconverted photons.

In summary, we have elucidated the spatial shape of entangled photon pairs generated by SPDC in different non-collinear geometries which also exhibit Poynting vector walk-off for some of the interacting waves. We found, in particular, that the OAM selection rule of entangled pairs generated in collinear SPDC, namely \( l_0 = l_1 + l_2 \), does not hold in general, even for nearly collinear geometries, when using strongly focused pump beams. Notwithstanding, under appropriate
The spatial shape of entangled photon states generated in non-collinear, walking parametric downconversion conditions that concern the geometric configuration of the SPDC process, the pump beam width, and the crystal length, it can be fulfilled. We notice that this is particularly important for the implementation of quantum information protocols based on spatially encoded information. The ellipticity of the spatial shape of the entangled photons is induced by the walk-off length of the pump beam, and by the non-collinear length. The importance of both effects is dictated by the length of the nonlinear crystal, and by the diffraction length of the pump beam, in sharp contrast to collinear phase matching with non-critical phase matching. We stress that the effects described here are directly relevant to current experiments \cite{1, 11, 12}.

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