# Shaping soliton properties in Mathieu lattices

**Yaroslav V. Kartashov, Alexey A. Egorov,* Victor A. Vysloukh,† and Lluis Torner**

*ICFO-Institut de Ciencies Fotoniques, and Universitat Politecnica de Catalunya, Barcelona 08034, Spain*

Received July 12, 2005; revised September 16, 2005; accepted October 10, 2005

We address basic properties and stability of two-dimensional solitons in photonic lattices induced by the nondiffracting Mathieu beams. Such lattices allow for smooth topological transformation of radially symmetric Bessel lattices into quasi-one-dimensional periodic ones. The transformation of lattice topology drastically affects the properties of ground-state and dipole-mode solitons, including their shape, stability, and transverse mobility. © 2006 Optical Society of America

*OCIS codes:* 190.5530, 190.4360, 060.1810.

Transverse variations of the refractive index of nonlinear media drastically alter the propagation of optical solitons, which can hence be routed and steered. The optical induction technique, recently introduced in nonlinear optics,[1–5] opens broad horizons for creation of various transverse refractive index landscapes, or optical lattices. Lattices with tunable features can thus be readily imprinted in suitable crystals. The basic properties of solitons supported by the lattice are defined by its topology. Thus the domain of soliton existence in the simplest periodic lattice, formed by a set of plane waves, is dictated by the Floquet–Bloch lattice spectrum.[1–10] Another important class of optical lattices can be created by nondiffracting Bessel beams with radial symmetry. Such lattice symmetry results in new soliton features and opens new ways of soliton manipulation, including the possibility of inducing rotary soliton motions and collisions in different lattice rings,[11–13] and the creation of reconfigurable soliton networks.[14,15]

In this Letter we study a new type of optical lattice that sets an important connection between periodic and radially symmetric Bessel lattices. Such lattices can be induced by nondiffracting Mathieu beams, and they afford smooth topological transformation of a radially symmetric profile of Bessel lattice into a quasi-one-dimensional periodic one. The transformation of the lattice topology finds its manifestation in a dramatic change of the properties of the ground-state and dipole-mode solitons.

We start our analysis with a generic equation describing the propagation of laser radiation along the $\xi$ axis of a Kerr-type cubic nonlinear medium with imprinted transverse modulation of the refractive index:

$$i\frac{\partial q}{\partial \xi} = -\frac{1}{2}\left(\frac{\partial^2 q}{\partial \eta^2} + \frac{\partial^2 q}{\partial \zeta^2}\right) + \sigma q|q|^2 - pR(\eta,\zeta)q. \qquad (1)$$

Here $q(\eta,\zeta,\xi)$ is the dimensionless amplitude of the light field; the longitudinal $\xi$ and the transverse $\eta,\zeta$ coordinates are scaled to the diffraction length and the input beam width, respectively; parameter $\sigma = \pm 1$ stands for the nonlinearity sign (defocusing–focusing); $p$ describes the lattice depth; and the function $R(\eta,\zeta)$ stands for the transverse lattice profile. We assume that the optical lattice features an intensity $R(\eta,\zeta) \sim |q_M|^2$ of the nondiffracting Mathieu beam, similarly to periodic lattices created in a photorefractive medium,[1–10] while $\max[R(\eta,\zeta)]=1$. The field distribution of the Mathieu beam can be written via a Whittaker integral[16–18]:

$$q_M(\eta,\zeta,\xi) = \exp(ib_{\text{lin}}\xi)\int_{-\infty}^{\infty} A(\phi)\exp[i(-2b_{\text{lin}})^{1/2} (\eta\cos\phi + \zeta\sin\phi)]\mathrm{d}\phi, \qquad (2)$$

where $b_{\text{lin}} < 0$ is the propagation constant; $A(\phi)$ is the angular spectrum represented by even $\text{ce}_m(\phi, -b_{\text{lin}}a^2/2)$, $m=0,1,2\ldots$, or odd $\text{se}_m(\phi,-b_{\text{lin}}a^2/2)$, $m=1,2,3\ldots$ angular Mathieu functions; and $a$ is the interfocal parameter. Mathieu beams are fundamental nondiffracting solutions of the wave equation in elliptical cylindrical coordinates. They can be generated by illumination of a narrow annular slit with a Gaussian aperture placed in the focal plane of a lens,[17,18] while the beam topology can be controlled by the width of the aperture. Equation (1) conserves the energy flow: $U=\iint_{-\infty}^{\infty}|q|^2\mathrm{d}\eta\mathrm{d}\zeta$.

Representative examples of the lowest-order even ($m=0$) and odd ($m=1$) Mathieu lattices considered in this Letter are shown in Fig. 1. For small values of the interfocal parameter, $a\to 0$, the foci of the associated elliptical coordinate system collapse to a point, and the even Mathieu lattice transforms into a radially symmetric Bessel lattice. Odd Mathieu beams produce azimuthally modulated lattices at $a\to 0$. At

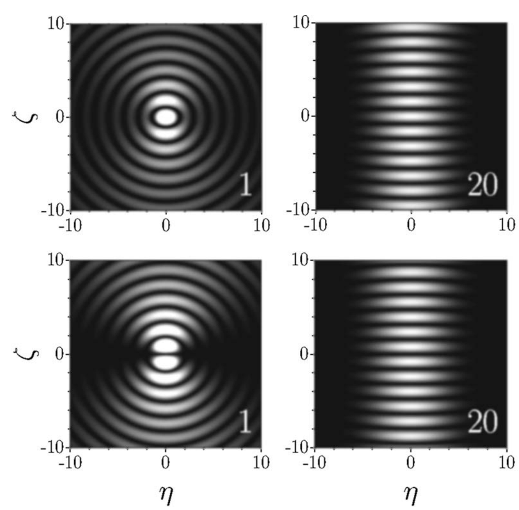


Fig. 1. Lowest-order even (top) and odd (bottom) Mathieu lattices with different values of interfocal parameter $a$.

 

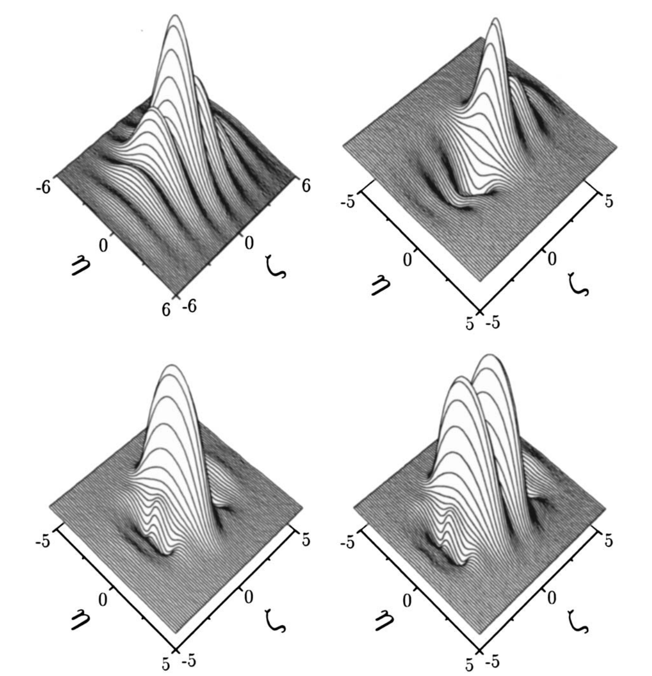


Fig. 2. Top row, ground-state soliton in an even lattice at $b=2.9$, $a=500$, $p=5$ (left) and dipole-mode soliton in an odd lattice at $b=2.84$, $a=30$, $p=5$ (right). Focusing medium. Bottom row, simplest gap soliton in an even lattice at $b=3.2$, $a=12$, $p=15$ (left) and combination of in-phase gap solitons in an odd lattice at $b=3.2$, $a=12$, $p=15$ (right). Defocusing medium.

$a\rightarrow\infty$ when separation of the foci tends to infinity, the lattice transforms into a quasi-one-dimensional periodic pattern. Thus, modification of the interfocal parameter results in a smooth topological deformation of lattice shape. The lattice frequency in the $\zeta$ direction is dictated by the parameter $b_{\text{lin}}$, so we further fix $b_{\text{lin}}=-2$ and vary $a$ and $p$.

Profiles of the simplest solitons supported by Mathieu lattices are depicted in Fig. 2. We found them in the form $q(\eta,\zeta,\xi)=w(\eta,\zeta)\exp(ib\xi)$, where $w(\eta,\zeta)$ is the real function and $b$ is the propagation constant. Ground-state solitons reside on the central maximum of the simplest ($m=0$) even Mathieu lattice imprinted in the focusing medium. At $a=0$ such solitons are radially symmetric. With an increase of $a$ at fixed $U$ and $p$, the solitons gradually become elliptical, their amplitude decreases, and they expand over neighboring lattice maxima along the $\zeta$ axis. For large $a$ values solitons may feature strong modulation along the $\zeta$ axis, whose depth increases with an increase of lattice depth. Mathieu lattices imprinted in a focusing medium support dipole-mode solitons whose field changes sign between two central lattice maxima (below we consider such solitons in odd lattices with $m=1$). Lattices in a defocusing medium support specific solitons that can be termed gap solitons. The localization mechanism for such solitons in the $\zeta$ direction (where the lattice is almost periodic for $a\gg 1$) is of the Bragg type, while confinement in $\eta$ is achieved because of appropriate refractive compensation of diffraction and defocusing. In-phase combinations of such gap solitons are also possible (Fig. 2).

Transformation of lattice topology substantially affects solitons' stability. The properties of ground-state solitons at $\sigma=-1$ are summarized in Fig. 3. At small values of $a$, the energy flow increases monotonically with $b$ [Fig. 3(a)]. The energy flow vanishes in the cutoff on $b$ that increases with lattice depth $p$ and interfocal parameter $a$ [Fig. 3(b)]. With an increase of $b$, solitons get narrower, their amplitude increases, and their energy flow asymptotically approaches $U_{\text{cr}}\approx 5.85$, which corresponds to the energy flow of a Townes soliton in a uniform medium. Thus, at $a=50$, $p=1$, and $b=2$ the soliton peak amplitude is 2.78 and $U\approx 5.23$, while at $b=7.5$ the peak amplitude reaches 6.22 and $U\approx 5.69$. At a given critical value of $a$, a branch with negative $\mathrm{d}U/\mathrm{d}b$ appears. According to the Vakhitov–Kolokolov (VK) criterion this branch corresponds to unstable solitons. Solitons belonging to this branch expand over many lattice periods in the $\zeta$ direction (Fig. 2), which results in an increase of $U$ with a decrease of $b$, so that the energy flow becomes a two-valued function of $b$. Such a soliton expansion in the small amplitude limit is consistent with the fact that linear guided modes of Mathieu lattices for large $a$ approach delocalized Bloch modes of quasi-one-dimensional periodic patterns. In contrast, at $a\ll 1$ low-amplitude solitons transform into linear guided modes of an almost radially symmetric lattice, which are always well localized. The instability domain for the ground-state soliton broadens with $a$ [Fig. 3(c)]. Direct stability analysis performed with Eq. (1), linearized around perturbed stationary solution $w(\eta,\zeta)$, confirmed conclusions based on the VK criterion. Instability of the ground-state soliton in the Mathieu lattice at $a\gg 1$ is associated with exponentially growing perturbation [Fig. 3(d)].

The topology of the Mathieu lattice has an even stronger effect on the properties of dipole-mode solitons. The energy flow of such solitons increases

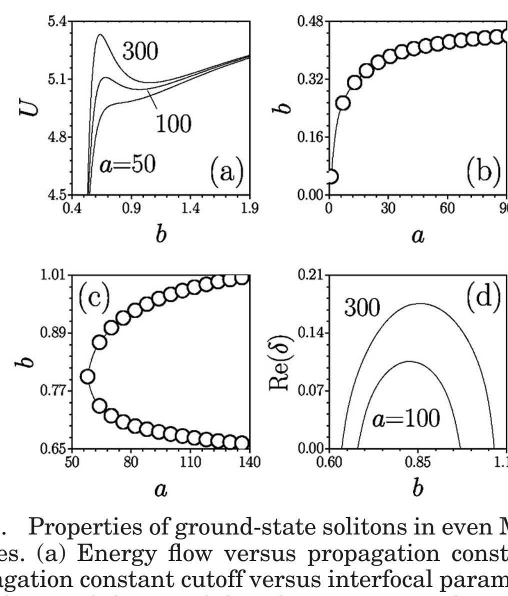


Fig. 3. Properties of ground-state solitons in even Mathieu lattices. (a) Energy flow versus propagation constant. (b) Propagation constant cutoff versus interfocal parameter. (c) Boundaries of the instability domain versus the interfocal parameter. (d) Real part of the perturbation growth rate versus the propagation constant. In all cases $p=1$. Focusing medium.

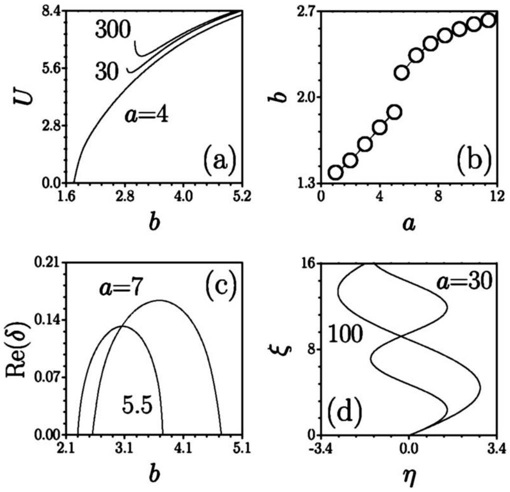


Fig. 4. Properties of dipole-mode solitons in odd Mathieu lattices. (a) Energy flow versus propagation constant. (b) Propagation constant cutoff versus the interfocal parameter. (c) Real part of the perturbation growth rate versus the propagation constant. (d) Trajectory of the center of a ground-state soliton with $b=4$ and input tilt $\alpha_\eta=1$ launched into even lattices corresponding to different $a$. In all cases $p=5$. Focusing medium.

monotonically with $b$ at small $a$ [Fig. 4(a)]. Energy flow vanishes in the cutoff, where solitons broaden substantially. For high enough $a$, the soliton ceases to exist in the cutoff without any topological shape transformation. This behavior finds its manifestation in the discontinuity of the dependence of cutoff on interfocal parameter [Fig. 4(b)]. With the growth of $b$ the dipole-mode soliton transforms into two narrow weakly interacting out-of-phase solitons. Stability analysis revealed that dipole-mode solitons became stable above a threshold on the propagation constant (or $U$), similarly to their counterparts in two-dimensional periodic lattices[8] [Fig. 4(c)]. Dipole-mode solitons suffer oscillatory instabilities associated with complex growth rates. Importantly, we found that the threshold value of propagation constant for stabilization quickly increases with $a$, upon transformation of the lattice into a quasi-one-dimensional one, so that at $a \to \infty$ dipole-mode solitons become unstable in the entire domain of their existence. This shows the drastic difference between the stability properties of dipole-mode solitons in quasi-one-dimensional and two-dimensional lattices, which always supports stable dipole-mode solitons. Gap solitons and their in-phase combinations depicted in Fig. 2 can also be made stable in Mathieu lattices.

Transformation of the lattice topology with an increase of $a$ substantially affects soliton mobility. Solitons are strongly pinned by almost radially symmetric lattices with small $a$ and can hardly jump into neighboring lattice rings. In contrast, quasi-one-dimensional lattices with large $a$ feature pronounced channels along the $\eta$ axis, so that even small input tilts cause considerable transverse displacements (periodic oscillations) of both ground-state and dipole-mode solitons [Fig. 4(d)]. The amplitude of these oscillations increases with $a$ for a given input tilt. Notice also the possibility of inducing rotary motions in lowest-order even Mathieu lattices with small interfocal parameters.

Summarizing, we analyzed the properties of solitons supported by Mathieu lattices that establish important connections between radially symmetric and quasi-one-dimensional periodic lattices. We showed how lattice topology affects the basic properties of ground-state and dipole-mode solitons, including their stability and mobility in the transverse plane.

Y. V. Kartashov (Yaroslav.Kartashov@icfo.es) and L. Torner acknowledge support of the Spanish government through grant BFM2002-2861 and the Ramon-y-Cajal program. V. A. Vysloukh acknowledges CONACyT support through project 46552.

*Visiting from the Department of Physics, M. V. Lomonosov Moscow State University, Moscow, Russia.

†Visiting from the Departamento de Fisica y Matematicas, Universidad de las Americas, Puebla, Mexico.

## References

1. N. K. Efremidis, S. Sears, D. N. Christodoulides, J. W. Fleischer, and M. Segev, Phys. Rev. E **66**, 046602 (2002).
2. J. W. Fleischer, T. Carmon, M. Segev, N. K. Efremidis, and D. N. Christodoulides, Phys. Rev. Lett. **90**, 023902 (2003).
3. J. W. Fleischer, M. Segev, N. K. Efremidis, and D. N. Christodoulides, Nature (London) **422**, 147 (2003).
4. D. Neshev, E. Ostrovskaya, Y. Kivshar, and W. Krolikowski, Opt. Lett. **28**, 710 (2003).
5. H. Martin, E. D. Eugenieva, Z. Chen, and D. N. Christodoulides, Phys. Rev. Lett. **92**, 123902 (2004).
6. J. Yang and Z. Musslimani, Opt. Lett. **28**, 2094 (2003).
7. Y. V. Kartashov, A. A. Egorov, L. Torner, and D. N. Christodoulides, Opt. Lett. **29**, 1918 (2004).
8. J. Yang, I. Makasyuk, A. Bezryadina, and Z. Chen, Opt. Lett. **29**, 1662 (2004).
9. D. N. Neshev, T. J. Alexander, E. A. Ostrovskaya, Y. S. Kivshar, H. Martin, I. Makasyuk, and Z. Chen, Phys. Rev. Lett. **92**, 123903 (2004).
10. J. W. Fleischer, G. Bartal, O. Cohen, O. Manela, M. Segev, J. Hudock, and D. N. Christodoulides, Phys. Rev. Lett. **92**, 123904 (2004).
11. Y. V. Kartashov, V. A. Vysloukh, and L. Torner, Phys. Rev. Lett. **93**, 093904 (2004).
12. Y. V. Kartashov, A. A. Egorov, V. A. Vysloukh, and L. Torner, Phys. Rev. E **70**, 065602(R) (2004).
13. Y. V. Kartashov, V. A. Vysloukh, and L. Torner, Opt. Lett. **30**, 637 (2005).
14. Z. Xu, Y. V. Kartashov, and L. Torner, Opt. Express **13**, 1774 (2005).
15. Z. Xu, Y. V. Kartashov, L. Torner, and V. A. Vysloukh, Opt. Lett. **30**, 1180 (2005).
16. J. C. Gutierrez-Vega, M. D. Iturbe-Castillo, and S. Chavez-Cerda, Opt. Lett. **25**, 1493 (2000).
17. J. C. Gutierrez-Vega, M. D. Iturbe-Castillo, G. A. Ramirez, E. Tepichin, R. M. Rodriguez-Dagnino, S. Chavez-Cerda, and G. H. C. New, Opt. Commun. **195**, 35 (2001).
18. S. Chavez-Cerda, M. J. Padgett, I. Allison, G. H. C. New, J. C. Gutierrez-Vega, A. T. O'Neil, I. MacVicar, and J. Courtial, J. Opt. B: Quantum Semiclassical Opt. **4**, S52 (2002).