Three-dimensional spatiotemporal optical solitons in nonlocal nonlinear media

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We demonstrate the existence of stable three-dimensional spatiotemporal solitons (STSs) in media with a nonlocal cubic nonlinearity. Fundamental (nonspinning) STSs forming one-parameter families are stable if their propagation constant exceeds a certain critical value that is inversely proportional to the range of nonlocality of nonlinear response. All spinning three-dimensional STSs are found to be unstable.

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Spatiotemporal solitons (STSs), also referred to as "light bullets" [1], have attracted a great deal of attention in optics; see the recent review [2]. These are multidimensional pulses, which maintain their shape in the longitudinal (temporal) and transverse (spatial) directions due to the balance between the group-velocity dispersion, diffraction, and nonlinear self-phase modulation. However, solitons in media with the cubic self-focusing nonlinearity, obeying the nonlinear Schrödinger (NLS) equation, are unstable in two and three dimensions (2D and 3D), because of the occurrence of collapse in the same model [3]. Several possibilities to arrest the collapse were considered, such as periodic alternation of focusing and defocusing layers [4] and various generalizations of this setting [5], and the use of media exhibiting saturable [6] or quadratic ($\chi^{(2)}$) [7] nonlinearities; tandem structures, composed of linear and nonlinear materials with engineered properties, were also proposed [8]. The only successful experiment in this field was the creation of quasi-(2+1)-dimensional STSs in bulk $\chi^{(2)}$ samples [2,9]. Other theoretically developed approaches use off-resonance two-level systems [10] and self-induced-transparency media [11].

Collapse does not occur either in $\chi^{(3)}$ media whose nonlinearity is nonlocal [12], therefore they may also give rise to stable solitons; see the review [13]. 2D spatial solitons stabilized by the nonlocality were observed in vapors [14] and lead glasses featuring strong thermal nonlinearity [15]; in the latter case, elliptic and vortex-ring solitons were reported. Optical 1D solitons supported by a nonlocal $\chi^{(3)}$ nonlinearity were also created in liquid crystals [16]. Further, self-focusing in photorefractive media [17], periodic lattices [18], vortices [19], and spatial solitons in soft matter [20] were considered in the context of nonlocality.

The objective of this work is to demonstrate that one-parameter families of *stable* (3+1)-dimensional STSs are possible in media with nonlocal $\chi^{(3)}$ nonlinearity. For this purpose, we consider a 3D model based on a general system of coupled equations for the complex field amplitude q and nonlinear correction to the refractive index n (see, e.g., [21]); in a normalized form, the equations are

$$iq_\xi + (1/2)(q_{\eta\eta} + q_{\zeta\zeta} + Dq_{\tau\tau}) + qn = 0,$$

$$d(n_{\eta\eta} + n_{\zeta\zeta}) - n + |q|^2 = 0. \tag{1}$$

Here, η, ζ and ξ are the transverse and longitudinal coordinates, scaled, respectively, to the beam's width and its diffraction length, τ is the reduced temporal variable, and D is the ratio of the diffraction and dispersion lengths. We consider the case of *anomalous* temporal dispersion, $D > 0$, and then set $D = 1$ by means of an obvious scaling. Lastly, $\sqrt{d}$ determines the correlation length Λ_{corr} of the nonlocal nonlinear response [note that by setting $d = 0$ one turns Eqs. (1) into the ordinary 3D NLS equation with self-focusing]. In fact, after setting $D = 1$, additional rescaling of Eqs. (1) makes it possible to set $d = 1$, so as to cast the system into the parameter-free form. Nevertheless, we prefer to keep d as an explicit parameter, as it directly controls the system's nonlocality degree.

The nonlocal nonlinearity in Eqs. (1) is typical for light propagation in liquid crystals and for thermal nonlinearity in optical media [13,16]. For the derivation of the model equations (1), the usual approximation of the slowly varying amplitude is adopted, along with an assumption of fast temporal relaxation of the refractive-index perturbations, therefore the second equation in (1) does not contain the term $n_{\tau\tau}$. The latter assumption, which is essential for the physical justification of the nonlocal model including the temporal-dispersion term, implies that the relaxation time, $\tau_{\text{rel}} \sim \Lambda_{\text{corr}}/c$ (c is the light speed), must be of the order of the pulse duration T of the STS to be constructed. Whether such conditions can be met in available optical media featuring nonlocal nonlinearities, such as liquid crystals, which tend to exhibit slower relaxation times, is a question that remains a challenge.

Equations (1) conserve the energy $E = \iiint |q(\eta, \zeta, \tau)|^2 d\eta d\zeta d\tau$, Hamiltonian H, the momentum $P_{\eta,\zeta}$ in the transverse plane, and angular momentum M_ξ along the longitudinal direction. Stationary solutions to Eqs. (1) are looked for as $q = w(r, \tau)\exp[i(b\xi + S\theta)]$, $n = n(r, \tau)$, where r and θ are the polar coordinates in the (η, ζ) plane, b is a real propagation constant, integer S is the vorticity ("spin"), and real functions w and n obey the equations
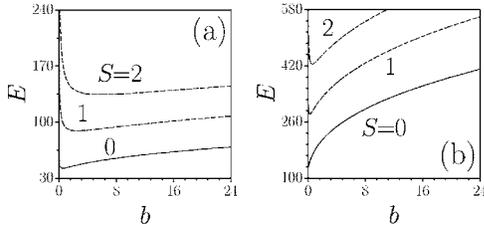
FIG. 1. Energy E vs propagation constant b for soliton families with different vorticities, $S=0$, 1, and 2 (numbers labeling the curves), at different values of the range of nonlocality of nonlinear response $\sqrt{d}$: (a) $d=1$, (b) $d=10$. Full and dashed lines show stable and unstable solitons.

$$(w_{rr} + r^{-1}w_r + w_{\tau\tau}) - (2b + r^{-2}S^2)w + 2wn = 0, \qquad (2)$$

$$d(n_{rr} + r^{-1}n_r) - n + w^2 = 0 \qquad (3)$$

(for this solution, the angular momentum is proportional to the energy, $M_\xi = SE$).

We have numerically found families of localized solutions to these equations, dealing with the corresponding two-point boundary-value problem by dint of the standard band-matrix algorithm. Typically, grids with 241×240 and 201×360 points were used for the computations of the fundamental ($S=0$) and spinning solitons, respectively. In Fig. 1, we display the energy characteristic, $E=E(b)$, for one-parameter families of the thus constructed soliton solutions. In view of the above-mentioned possibility to eliminate d by means of rescaling, panel (a) in Fig. 1 is, as a matter of fact, a blowup of a part of panel (b) corresponding to $0 < b < 2.4$. A noteworthy feature is that the 3D solitons exist only above a finite energy threshold.

The stabilizing effect of the nonlocality for the fundamental solitons is seen in Fig. 1: except for a narrow interval of small wave numbers, $0 < b < b_{cr}$, the solitons are expected to be stable, as they satisfy the *Vakhitov-Kolokolov* (VK) criterion, $dE/db > 0$, which is a necessary (but, generally, not sufficient) stability condition for the soliton family [3,22] (note that the instability of the 3D solitons in the local NLS equation precisely follows this criterion). The bending of the curve $b = b(E)$ for fundamental 3D solitons, from the negative to positive slope, was also found in a different model (a $\chi^{(3)}$ medium with a Gaussian nonlocal kernel) by means of a variational approximation [23]. Below, it will be demonstrated that the VK criterion is actually sufficient for the stability of the fundamental solitons, but not for spinning ones, which are always unstable.

For $d=1$, the critical wave number which separates the negative- and positive-slope regions for the fundamental solitons in Fig. 1 is $b_{cr}^{(S=0)} = 0.565$, and, in view of the scaling invariance of the model, $b_{cr}^{(S=0)}(d) = 0.565/d$. The energy of the $S=0$ soliton at the critical point is $E_{cr}^{(S=0)} = 42.60$ for $d=1$ (in the model with $d \neq 1$, E_{cr} scales as $\sqrt{d}$, see Fig. 1). It also follows from here that $E_{cr}^{(S=0)} = 32.02(b_{cr}^{(S=0)})^{-1/2}$, which may be compared to the scaling law for unstable fundamental solitons in the 3D local NLS equation, $E^{(S=0)} = C_0 b^{-1/2}$, with $C_0 \approx 6.67$ [3].
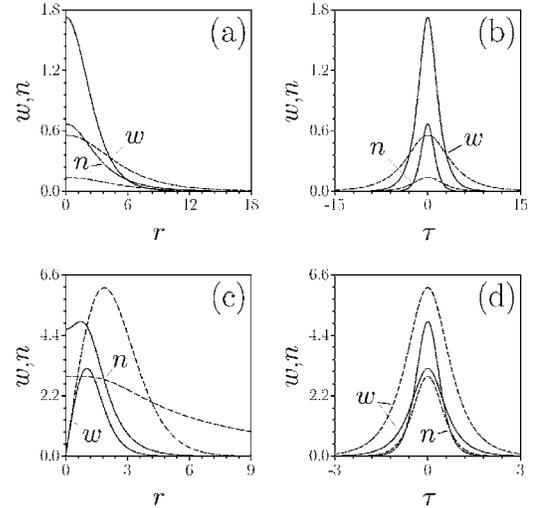

FIG. 2. Cross-section shapes of typical $S=0$ and $S=1$ solitons in the transverse (r) and temporal (τ) directions. (a) and (b) $S=0$, $d=10$, $E=140$; full and dashed lines correspond to stable and unstable solitons, with $b=0.15$ and $b=0.021$, respectively. (c) and (d) $S=1$, $b=1$; solid and dashed lines correspond to $d=1$ and $d=100$, respectively (due to the scaling invariance, the latter is tantamount to $d=1$ and $b=100$).

Figures 2(a) and 2(b) display the shapes of typical stable and unstable fundamental soliton for $d=10$, at a fixed value of the energy ($E=140$). It is seen that high-amplitude solitons are stable, whereas low-amplitude ones are unstable. We notice that both the low- and high-amplitude solitons feature spatiotemporal ellipticity, being broader in space than in time, which can be easily explained by a perturbative analysis of Eqs. (1) for small d. Typical shapes of unstable $S=1$ solitons are shown in Figs. 2(c) and 2(d).

Full stability of solitons was investigated using the equations for small perturbations linearized around the stationary solution. Accordingly, solutions including perturbations with an infinitesimal amplitude ϵ are looked for as

$$q = e^{ib\xi + iS\theta}\{w(r,\tau) + \epsilon[f(r,\tau)e^{\delta\xi + iJ\theta} + g^*(r,\tau)e^{\delta^*\xi - iJ\theta}]\},$$
$$(4)$$

$$n = n(r,\tau) + \epsilon[p(r,\tau)e^{\delta\xi + iJ\theta} + p^*(r,\tau)e^{\delta^*\xi - iJ\theta}], \qquad (5)$$

where J is an arbitrary integer azimuthal index of the perturbation, δ is the instability growth rate, the asterisk stands for the complex conjugation, and the eigenfunctions f, g and p, obey the equations

$$2i\delta f + f_{rr} + r^{-1}f_r + f_{\tau\tau} - [2b + (S+J)^2 r^{-2}]f + 2(nf + wp) = 0,$$

$$-2i\delta g + g_{rr} + r^{-1}g_r + g_{\tau\tau} - [2b + (S-J)^2 r^{-2}]g + 2(ng + wp)$$
$$= 0,$$

$$d(p_{rr} + r^{-1}p_r) - (1 + dJ^2 r^{-2})p + w(f + g) = 0. \qquad (6)$$

The growth rate δ was found as an eigenvalue at which Eqs. (6) has a nonsingular localized solution. In Figs. 3(a)–3(d), we plot the instability gain, Re(δ), versus the
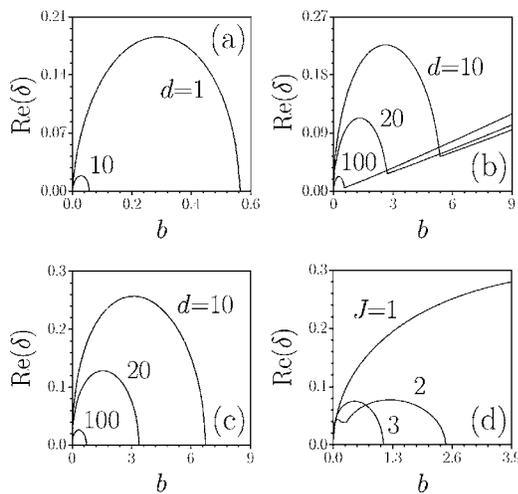
FIG. 3. The real part of the perturbation growth rate vs the propagation constant of the unperturbed soliton. (a) $S=0$, $J=0$, for $d=1$ and $d=10$; (b) $S=1$, $J=1$; (c) $S=1$, $J=2$; (d) $S=2$, $d=20$. In (b) and (c), values of d, and in (d), values of J are indicated near the curves.

propagation constant b of the unperturbed solution, for the STSs with $S=0$, 1, and 2, respectively. The stable solitons are those for which Re$(\delta)=0$ for all (integer) values of J. Due to the scaling invariance of Eqs. (1), the curves in a given panel, pertaining to different values of d, are obtained from each other by the scaling transformation, therefore they actually display essentially the same stability and instability regions, but on different scales. Figure 3(a) shows Re(δ) for $J=0$, as $J\neq0$ does not yield any instability for the fundamental solitons; the stability region revealed by this figure, $b>b_{cr}$, is precisely the same as predicted above by the VK criterion. For the spinning solitons, the perturbations with $J>1$ destabilize a part of the families, see, e.g., Fig. 3(c) for $S=1$ and $J=2$, but entire STS families with $S\geq1$ are unstable against the perturbations with $J=1$, see Figs. 3(b) and 3(d). The latter instability mode implies a trend to splitting of the vortex soliton into a set of two fundamental ones [2], which is corroborated by direct simulations below.
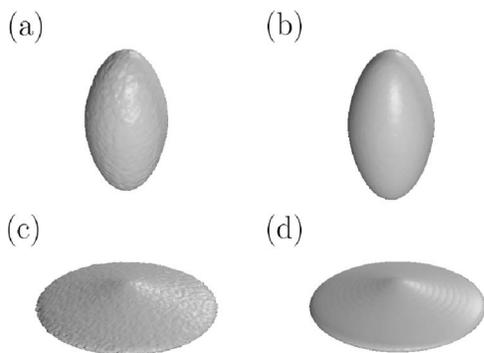

FIG. 4. Isosurface plots illustrating the stability of a fundamental soliton corresponding to $d=10$, $b=1$, and $E=178$. (a) and (c) The initially perturbed soliton, at $\xi=0$; (b) and (d) the self-cleaned one at $\xi=360$. Here and in Figs. 5 and 6, the upper and lower rows show $|q|^2$ and n, respectively.
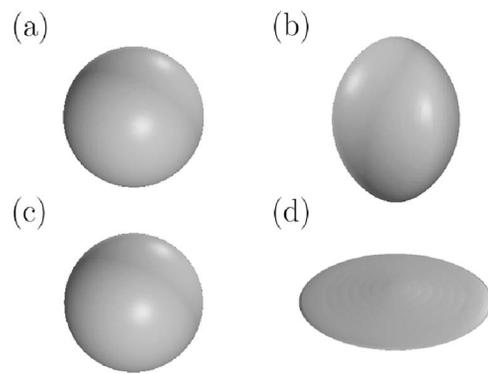

FIG. 5. Self-trapping of a fundamental soliton, for $d=1$: (a) and (c) the initial Gaussian pulse with energy $E_0=54$; (b) and (d) the soliton at $\xi=60$.

Note that *stable* 3D spinning optical solitons (with $S=1$) were only found in media with *competing* focusing and defocusing nonlinearities, viz., $\chi^{(3)}:\chi^{(5)}$ or $\chi^{(2)}:\chi^{(3)}$ [24]. On the other hand, stable 2D spatial (rather than spatiotemporal) vortex rings were experimentally observed in a medium featuring the thermal nonlocal nonlinearity [15].

The predictions of the linear stability analysis were checked in direct simulations of Eqs. (1), which were run by means of a standard Crank-Nicholson scheme. The nonlinear finite-difference equations were solved using the Picard iteration method, and the resulting linear system was handled with the help of the Gauss-Seidel iterative procedure. To achieve good convergence, we needed typically twelve Picard and four Gauss-Seidel iterations. The initial conditions for perturbed solitons were taken as $q(\xi=0)=w(\eta,\zeta,\tau)(1+\epsilon\rho)$ and $n(\xi=0)=n(\eta,\zeta,\tau)(1+\epsilon\rho)$, where ϵ is, as above, a small perturbation amplitude, and ρ was either a random variable uniformly distributed in the interval $[-0.5,0.5]$, or simply $\rho=1$ (uniform perturbation).

First, we have checked that all the fundamental STSs that were predicted above to be stable (stable branches are shown with solid lines in Fig. 1) are stable indeed against random perturbations; Fig. 4 displays an example of self-healing of a stable soliton with the initial perturbation amplitude $\epsilon=0.1$. A small uniform perturbation ($\rho=1$) applied to a stable soliton excites its persistent oscillations, which suggests the existence of a stable intrinsic mode in the soliton. On the other hand, direct simulations show that those fundamental
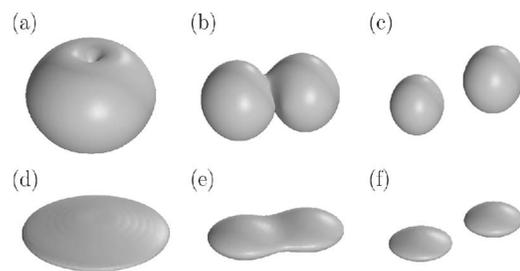

FIG. 6. Splitting of an unstable $S=1$ soliton with $d=100$ and $b=0.05$. (a) and (d) $\xi=0$, (b) and (e) $\xi=1400$, and (c) and (f) $\xi=1600$.

solitons that were predicted to be unstable decay into radiation, if slightly perturbed. We also simulated self-trapping of a stable fundamental soliton from an initial spatiotemporal pulse of an arbitrary form. An example is shown in Fig. 5 for an isotropic Gaussian input, which generates an anisotropic (elliptic) soliton. We also simulated the evolution of unstable spinning solitons. Most typically, they follow the prediction of the linear stability analysis and split into two stable fundamental solitons; see an example (for $S=1$) in Fig. 6.

In conclusion, we have demonstrated that nonlocal cubic nonlinearity is sufficient to stabilize 3D solitons, which suggests a new approach to making 3D spatiotemporal solitons in optics, which thus far evaded experimental observation. The stability of the fundamental solitons was demonstrated

through the computation of the corresponding stability eigenvalues, and in direct simulations. Their robustness and, hence, physical relevance were demonstrated by self-trapping from arbitrary input pulses. On the other hand, all the spinning 3D solitons in the nonlocal medium are unstable against splitting into a set of stable fundamental solitons.

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