Capacities of noiseless quantum channels for massive indistinguishable particles: Bosons versus fermions

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We consider information transmission through a noiseless quantum channel, where the information is encoded into massive indistinguishable particles: bosons or fermions. We study the situation in which the particles are noninteracting. The encoding input states obey a set of physically motivated constraints on the mean values of the energy and particle number. In such a case, the determination of both classical and quantum capacity reduces to a constrained maximization of entropy. In the case of noninteracting bosons, signatures of Bose-Einstein condensation can be observed in the behavior of the capacity. A major motivation for these considerations is to compare the information-carrying capacities of channels that carry bosons with those that carry fermions. We show analytically that fermions generally provide higher channel capacity, i.e., they are better suited for transferring bits as well as qubits, in comparison to bosons. This holds for a large range of power-law potentials, and for moderate to high temperatures. Numerical simulations seem to indicate that the result holds for all temperatures. Also, we consider the low-temperature behavior for the three-dimensional box and harmonic trap, and again we show that the fermionic capacity is higher than the bosonic one for sufficiently low temperatures.

I. INTRODUCTION

The capacity of a transmission channel is an important quantity in several aspects, ranging from fundamental theoretical issues to technology; it is therefore intensively studied in classical [1] and quantum [2] information theory. The capacity of a channel depends on different factors, such as the type of information that one wants to transmit, the character of the physical material that realizes the transfer, the constraints that exist on the channel, the assistance that is available for the information transfer, etc.

The capacity of a channel for the transmission of classical information by using quantum states is an important example, which has attracted a lot of interest recently. In this respect, a fundamental result, obtained more than 30 years ago, is the "Holevo bound" [3,4] (see also [5–7]). An essential message conveyed by the Holevo bound is that *at most n bits (binary digits) of classical information can be carried by a quantum system of n distinguishable qubits (two-dimensional quantum systems)*. However, the Holevo bound includes arbitrary encoding and decoding strategies. This has the consequence that for information transmission with infinite dimensional systems through noiseless channels, e.g., with the modes of an electromagnetic field, the Holevo bound predicts infinite capacities.

A similar situation appears also for the capacity of transmitting quantum information, merely by definition.

To avoid such nonphysical values, it is important to introduce the relevant physical constraints, while maximizing over the classical or quantum information that can be encoded in a physical system. Classical capacities of channels that carry photons in the form of modes of an electromag-

netic field have been studied extensively in the past decade [5,8,9]. In this case, as well as for the quantum capacity, as considered in this paper, the natural choice is an energy constraint on the input states of the channel.

In this paper, we will deal with channels carrying particles that have a nonvanishing mass. Such a problem is motivated by recent experiments producing atomic waveguides in optical microstructures [10], or in atom chips [11]. These experiments indicate a new possibility of using massive particles in quantum channels for classical as well as quantum communication over macroscopic, or at least mesoscopic, distances. For such massive particles, it is natural to put an average particle number constraint along with an average energy constraint on the input state of the channel. We will show that with these constraints, the grand canonical ensemble of statistical mechanics attains the classical as well as quantum capacity of noiseless channels that carry trapped noninteracting massive particles. Moreover, we show that the classical and quantum capacities in the case of noiseless quantum channels (with the above physical constraints) are the same. Therefore, in the remainder of this paper, the phrase "capacity of (quantum) channels carrying massive particles" (usually implicitly implying that the constraints on energy and particle numbers are already taken into account), unless with the specific adjective, will mean the classical as well the quantum capacity. We should mention here that sometimes it is more natural to consider the average energy constraint with a fixed number of particles. In this case, the capacity is attained in the canonical ensemble. Calculations of different properties of the canonical ensemble are usually very hard, as compared to the grand canonical ensemble. However, several studies indicate that the two ensembles give similar val-

ues for the average occupation numbers even for a finite but moderately large number of particles [12]. However, there are physical quantities of interest that give drastically different values. For example, the fluctuation in the average number of particles in a Bose-Einstein condensate is unphysically large according to the grand canonical ensemble, while that in the canonical ensemble is physically meaningful [12]. In this paper, we will restrict ourselves to the grand canonical ensembles for finding the capacities of channels carrying massive particles. The capacities, being dependent only on the average occupation numbers, will be similar already for a moderately large number of particles.

Noninteracting bosons exhibit a critical phenomenon, namely Bose-Einstein condensation (BEC). In this paper, we will show that the capacity of channels carrying noninteracting bosons also exhibits this criticality with temperature. The capacity changes its behavior from being concave to convex (with respect to temperature) at the critical temperature. Noninteracting fermions do not show any critical behavior. However, interacting fermions can exhibit the Bardeen-Cooper-Schrieffer (BCS) transition. As we have shown in Ref. [13], the channel capacity in this case also indicates the BCS transition. It should be noted here that the convexity and concavity that we speak about in this paper are always with respect to temperature, and not with respect to mixing of states.

An important motivation of this study is to compare the capacities of channels carrying bosons with that of fermions. We will obtain the following result analytically:

Capacities of noiseless quantum channels carrying noninteracting spinless fermions are higher than that of spinless bosons for a wide range of power-law potentials for sufficiently high temperatures.

Since we obtain the above result by using perturbation theory up to the third order, it holds already for moderate temperatures. Note that our numerical simulations indicate that fermions are better carriers of information than bosons even in the low-temerature region.

It may be noted that although the quantum channel that we consider in the paper is noiseless, the states that are used to encode the information are allowed to be noisy (mixed states encoding). In the case of classical capacity, it will turn out that there exist ensembles of pure states, which can be used to encode the classical information (to be transferred) for attaining the maximal information transfer.

The paper is organized as follows. In Sec. II, we discuss the Holevo bound on classical information transmission. We define the classical capacity of quantum channels in the next section (Sec. III). In Sec. IV, we define the quantum capacity of quantum channels. In Sec. V, we discuss the important ensembles in statistical mechanics, and in its first three subsections, we discuss, respectively, the microcanonical, the canonical, and the grand canonical ensembles. In Sec. V D, we consider the situation in which there is a constraint on the average number of particles, while there is *no* constraint on the (exact or average) number of particles. In the next section (Sec. VI), we discuss the capacities of quantum channels with constrained inputs. We then briefly discuss the capacities of channels carrying photons in Sec. VII. In Sec. VIII, we consider spinless noninteracting bosons and fermions,

and calculate the capacity of channels in these cases. In Sec. IX, we prove a theorem stating that fermions can carry more information (classical and quantum) than bosons. In Sec. X, we consider the low-temperature case for the 3D box and the 3D harmonic trap. For both these cases, the fermionic capacity is again higher than the bosonic one for sufficiently low temperatures. In the last section (Sec. XI), we make some concluding remarks. This paper presents on the one hand the results of Ref. [13] with more details, and on the other hand generalizes them to the case of trapped noninteracting bosonic and fermionic gases for a large class of power-law potentials. Let us note that many such potentials are feasible with currently available technology. We will see that the signature of BEC in the channel capacity in the case of a harmonic trap is more pronounced than that in the case of a uniform trap, as considered in Ref. [13].

It may be worthwhile to mention here that the mathematics required for obtaining the capacities is similar to that in quantum statistical mechanics, where one maximizes the entropy under different constraints. This is not very surprising, as the bounds on information transfer, discussed in Secs. II and IV, are "entropylike" quantities.

II. THE HOLEVO BOUND

The Holevo bound is an upper bound on the amount of classical information that can be accessed from a quantum ensemble in which the information is encoded. Suppose, therefore, that a sender Alice (A) obtains the classical message i, and she knows that this happens with probability p_i. She wants to send it to a receiver Bob (B). To do so, Alice encodes the information i in a quantum state ρ_i, and sends the quantum state to Bob. Bob receives the ensemble $\{p_i, \rho_i\}$, and wants to obtain as much information as possible about i, for which he performs a measurement that gives the result m, with probability q_m. Let the corresponding post-measurement ensemble be $\{p_{i|m}, \rho_{i|m}\}$. The classical information gathered can be quantified by the mutual information between the message index i and the measurement outcome [1],

$$I(i:m) = H(\{p_i\}) - \sum_m q_m H(\{p_{i|m}\}). \tag{1}$$

Here $H(\{r_x\}) = -\Sigma_x r_x \log_2 r_x$ is the Shannon entropy of the probability distribution $\{r_x\}$. Throughout the paper, we calculate all the quantities on amounts of information transfer in bits (binary digits). Note that the mutual information can be seen as the difference between the initial disorder and the (average) final disorder. Bob will be interested to obtain the maximal information, which is the maximum of $I(i:m)$ over all measurement strategies. This quantity is called the accessible information,

$$I_{\text{acc}} = \max I(i:m), \tag{2}$$

where the maximization is over all measurement strategies.

The maximization involved in the definition of accessible information is usually hard to compute, and hence it is important to know bounds on I_{acc} [3,4,14]. In particular, in Ref. [3,4], a universal upper bound, called the Holevo bound (or

Holevo quantity), on I_{acc} is given (see also [5–7]),

$$I_{acc}(\{p_i, \rho_i\}) \leq \chi(\{p_i, \rho_i\}) \equiv S(\bar{\rho}) - \sum_i p_i S(\rho_i). \quad (3)$$

Here $\bar{\rho} = \Sigma_i p_i \rho_i$ is the average ensemble state, and

$$S(\varsigma) = -\operatorname{tr}(\varsigma \log_2 \varsigma) \quad (4)$$

is the von Neumann entropy of ς.

III. CLASSICAL CAPACITY OF A QUANTUM CHANNEL: UNCONSTRAINED INPUTS

Consider a quantum channel $\mathcal{R}$ that acts on d-dimensional quantum systems as inputs. Suppose that Alice wants to send some classical information i, that occurs with probability p_i, through this quantum channel to Bob. She encodes this classical information in the quantum state ρ_i, where the Hilbert space corresponding to the quantum states ρ_i is d-dimensional. The classical capacity of this quantum channel is the maximal classical information that can be sent through this channel, and is therefore the accessible information of the ensemble $\{p_i, \mathcal{R}(\rho_i)\}$, maximized over all such ensembles on the d-dimensional Hilbert space $\mathcal{H}^d$,

$$\mathcal{C} = \max_{\{p_i, \rho_i\} \text{ on } \mathcal{H}^d} I_{acc}(\{p_i, \mathcal{R}(\rho_i)\}). \quad (5)$$

However, capacities are usually defined in an asymptotic sense. Therefore, the classical capacity in this case is

$$\mathcal{C}^\infty = \lim_{n \to \infty} \frac{1}{n} \max_{\{p_i, \rho_i\} \text{ on } (\mathcal{H}^d)^{\otimes n}} I_{acc}(\{p_i, \mathcal{R}(\rho_i)\}). \quad (6)$$

The Holevo bound implies that

$$\mathcal{C} \leq \max_{\{p_i, \rho_i\} \text{ on } \mathcal{H}^d} \left[S(\mathcal{R}(\bar{\rho})) - \sum_i p_i S(\mathcal{R}(\rho_i)) \right] \quad (7)$$

and

$$\mathcal{C}^\infty \leq \lim_{n \to \infty} \frac{1}{n} \max_{\{p_i, \rho_i\} \text{ on } (\mathcal{H}^d)^{\otimes n}} \left[S(\mathcal{R}(\bar{\rho})) - \sum_i p_i S(\mathcal{R}(\rho_i)) \right]. \quad (8)$$

The quantity $\mathcal{C}^\infty$ is usually very difficult to handle. The Holevo-Schumacher-Westmoreland theorem[15,16] states that in the particular case in which the inputs are products on the tensor product Hilbert space $(\mathcal{H}^d)^{\otimes n}$, the capacity (let us denote the capacity in this case by $\mathcal{C}_d^{\infty,1}$) is given by

$$\mathcal{C}^{\infty,1} = \max_{\{p_i, \rho_i\} \text{ on } \mathcal{H}^d} \left[S(\mathcal{R}(\bar{\rho})) - \sum_i p_i S(\mathcal{R}(\rho_i)) \right]. \quad (9)$$

For the case of the noiseless quantum channel that carries d-dimensional quantum states noiselessly, all the above capacities equal to $\log_2 d$. This value is attained by any complete orthogonal basis of pure states on $\mathcal{H}^d$.

Now for infinite dimensional quantum systems, the channel capacity obtained in this way predicts an unphysical infinite capacity. This is because the Holevo bound itself does not include any constraint on the available physical resources

in an actual implementation of the information transfer. In particular, arbitrary encoding and decoding schemes are allowed. To avoid this infinite capacity, one usually maximizes the accessible information over all ensembles that satisfies certain physical constraints. Due to the form of the Holevo bound on accessible information, such constrained maximizations are very similar to the ones in statistical mechanics. The same is true for the case of the quantum capacity, which we briefly discuss in the succeeding section; in Sec. V, we briefly discuss some similar constrained maximizations of statistical mechanics.

IV. QUANTUM CAPACITY OF A QUANTUM CHANNEL: UNCONSTRAINED INPUTS

We now consider the case of sending qubits (as opposed to bits) using quantum channels. The quantum capacity can be considered in (at least) the following four different situations [17]: the quantum channel $\mathcal{R}$ (acting on d dimensional quantum states at its input) (i) without the help of additional classical communication (in this case, we call the quantum capacity Q^0), (ii) with an arbitrary amount of forward classical communication ($Q^\rightarrow$), (iii) with an arbitrary amount of backward classical communication ($Q^\leftarrow$), and (iv) with an arbitrary amount of both-way classical communication ($Q^\leftrightarrow$).

Let us define the first case, the other definitions being similar. So, the quantum capacity Q^0 is [18]

$$Q^0 = \sup \lim_{n \to \infty} \frac{\log_2 D(n)}{n}, \quad (10)$$

where the supremum is over all such cases when there exists a $D(n)$-dimensional subspace $\mathbb{S}(n)$, of the total input space $(\mathcal{H}^d)^{\otimes n}$, satisfying the average fidelity criterion

$$\lim_{n \to \infty} \int_{|\psi\rangle \in \mathbb{S}(n)} \langle \psi | \mathcal{R}^{\otimes n}(|\psi\rangle\langle\psi|) |\psi\rangle d|\psi\rangle = 1, \quad (11)$$

where no classical communication was used in transferring the input state from the sender to the receiver.

In the case of a noiseless channel, the fidelity criterion is automatically satisfied, and all the quantum capacities are equal to $\log_2 d$. Now, if the dimension d of the input space is infinite, the capacities are again infinite, as in the case of classical capacity.

There are several remarkable results that are known in the noisy case. In particular, Q^0 is the maximum coherent information [18–21],

$$Q^0 = \max_\rho [S(\mathcal{R}(\rho)) - S(\mathcal{I} \otimes \mathcal{R}(\Phi_\rho))], \quad (12)$$

where the maximization is over all quantum states ρ defined on the Hilbert space $\mathcal{H}^d$, Φ_ρ is a purification of ρ, and $\mathcal{I}$ is the identity operator acting on quantum states of the ancillary Hilbert space that is required for the purification. Furthermore [22,23],

$$Q^0 = Q^\rightarrow. \quad (13)$$

V. THE FUNDAMENTAL ENSEMBLES IN STATISTICAL MECHANICS

In this section, we discuss the fundamental ensembles in statistical mechanics (see, e.g., [24–26]).

A. The microcanonical ensemble

Suppose that we have a physical system described by the Hamiltonian $\mathcal{H}$. We assume that the system has a fixed particle number N, and a fixed energy E. (To be more general, one must allow for fluctuations around E and N.) This represents a closed and isolated system. We want to find the state ϱ_{MC} of the system such that it maximizes the von Neumann entropy S. Let $|E,N,k\rangle$ denote the state a energy E and particle number N, and where k enumerates the degeneracy. Let $\Omega(E,N)$ be the total number of orthogonal states with energy E and particle number N, so that $k = 1, 2, \ldots, \Omega(E,N)$. Since the system is isolated (as it has fixed energy and fixed number of particles), the set $\{|E,N,k\rangle\}_{k=1}^{\Omega(E,N)}$ spans the allowed Hilbert space. Consequently, the maximum entropy is reached by the state

$$\varrho_{\mathrm{MC}} = \frac{1}{\Omega(E,N)} \sum_{k=1}^{\Omega(E,N)} |E,N,k\rangle\langle E,N,k|, \qquad (14)$$

which is actually the identity on the allowed Hilbert space. This is the microcanonical state of the system, and the ensemble $\{1/\Omega(E,N), |E,N,k\rangle\}_{k=1}^{\Omega(E,N)}$ is called the microcanonical ensemble.

B. The canonical ensemble

Consider next a physical system, described by the Hamiltonian $\mathcal{H}$, which has a fixed particle number N and a fixed *average* energy E. The average energy constraint forces every state ϱ to follow

$$\mathrm{tr}(\mathcal{H}\varrho) = E. \qquad (15)$$

This physical system represents a closed, but not isolated, system. We again want to find the state ϱ_C of the system such that it maximizes the von Neumann entropy S. One finds that [24]

$$\varrho_C = \frac{1}{\mathcal{Z}_C} \exp(-\beta\mathcal{H}), \qquad (16)$$

the canonical state of the system. Here $\beta = 1/k_B T$, with k_B being the Boltzmann constant and T the absolute temperature. The canonical partition function $\mathcal{Z}_C$ is given by

$$\mathcal{Z}_C = \mathrm{tr}[\exp(-\beta\mathcal{H})]. \qquad (17)$$

The state ϱ_C is called the canonical state of the system and the ensemble $\{\frac{1}{\mathcal{Z}_C} \exp(-\beta\epsilon_i), |\epsilon_i\rangle\}$ is called the canonical ensemble, where $\{\epsilon_i, |\epsilon_i\rangle\}$ is the eigensystem of the Hamiltonian $\mathcal{H}$. Just like for the microcanonical ensemble, the particle number enters the calculations via the Hamiltonian and for determining the allowed Hilbert space. For example, in the calculation of the trace in Eq. (17), the summation runs over all combinations of different numbers of particles at different energy levels, under the constraint that the sum of all particles in all the levels is N. Moreover, for indistinguishable particles, we must also take care about the statistics of the particles. For example, if we have a trap, whose energy levels are ϵ_i, and in which N noninteracting spinless bosons are trapped, so that the bosons are described by the Hamiltonian $\mathcal{H}_b = \Sigma_i \epsilon_i a_i^\dagger a_i$ (where $a_i^\dagger$ and a_i are creation and destruction operators of the ith mode), we have

$$\mathcal{Z}_C^b = \underbrace{\sum_{n_0=0,1,\ldots} \sum_{n_1=0,1,\ldots} \cdots}_{\Sigma_i n_i = N} \exp\left(-\beta\sum_i \epsilon_i n_i\right) \qquad (18)$$

$$= \underbrace{\sum_{n_0=0,1,\ldots} \sum_{n_1=0,1,\ldots} \cdots}_{\Sigma_i n_i = N} \prod_i \exp(-\beta\epsilon_i n_i), \qquad (19)$$

where n_i is the number of particles at energy level ϵ_i. For a given value of energy E, the temperature T is given by $E = -\frac{\partial}{\partial\beta}(\log_e \mathcal{Z}_C)$.

C. The grand canonical ensemble

The next step is to consider a physical system, described by the Hamiltonian $\mathcal{H}$, which has a fixed *average* particle number N and a fixed *average* energy E. The average energy constraint forces every state ϱ to follow Eq. (15), while the average particle number constraint reads

$$\mathrm{tr}(\mathcal{N}\varrho) = N, \qquad (20)$$

where $\mathcal{N}$ represents the total particle number operator. For example, for a system of spinless bosons, $\mathcal{N} = \Sigma_i a_i^\dagger a_i$, where $a_i^\dagger$ and a_i are the creation and annihilation operators of the ith mode. This physical system represents an open system. We again want to find the state ϱ_{GC} of the system such that it maximizes the von Neumann entropy S. One finds that the state is [24]

$$\varrho_{\mathrm{GC}} = \frac{1}{\mathcal{Z}_{\mathrm{GC}}} \exp[-\beta(\mathcal{H} - \mu\mathcal{N})], \qquad (21)$$

i.e., the grand canonical state of the system. Here μ is the chemical potential. The grand canonical partition function $\mathcal{Z}_{\mathrm{GC}}$ is given by

$$\mathcal{Z}_{\mathrm{GC}} = \mathrm{tr}\{\exp[-\beta(\mathcal{H} - \mu\mathcal{N})]\}. \qquad (22)$$

Note that in this case, the number of particles is not fixed, and in particular,

$$\mathcal{Z}_{\mathrm{GC}}^b = \sum_{n_0^b=0}^{\infty} \sum_{n_1^b=0}^{\infty} \cdots \exp\left(-\beta\sum_i (\epsilon_i - \mu^b)n_i^b\right)$$

$$= \prod_i \frac{1}{1 - e^{-\beta(\epsilon_i - \mu^b)}}, \qquad (23)$$

in the case of noninteracting spinless bosons in a trap of energy levels $\{\epsilon_i\}$, and where we have now denoted the chemical potential by μ^b. The ensemble

$$\left\{ \frac{1}{\mathcal{Z}_{GC}^b} \exp\left(-\beta \sum_i (\epsilon_i - \mu^b) n_i^b\right), |n_0^b, n_1^b, \ldots, n_i^b, \ldots\rangle \right\}$$

(24)

is the grand canonical ensemble, where the elements of the ensemble runs over all combinations of the n_i^b's ($n_i^b = 0, 1, 2, \ldots, \infty$).

As opposed to the case of the canonical partition function, the grand canonical partition function requires the value of the chemical potential. We can determine the chemical potential from the constraint on the average particle number constraint, for given values of the temperature and average particle number. We can then find the grand partition function with these values. The average occupation numbers, in the case of noninteracting spinless bosons, are given by

$$\overline{n_i^b} = \frac{1}{e^{\beta(\epsilon_i - \mu^b)} - 1}.$$

(25)

The average energy is given by $E = \sum_i \overline{n_i^b} \epsilon_i$. Otherwise, for given values of the average energy and average particle number, the dual constraints on average energy and particle number can be simultaneously solved to obtain the chemical potential and the temperature.

D. The constraint on the number of particles

One may now consider the case in which we do not have a constraint on the number of particles. In other words, arbitrary numbers of particles are allowed. For the energy, we still assume that the average energy is fixed to E. In this case, maximizing the entropy leads to the partition function

$$\mathcal{Z}^b = \sum_{n_0=0}^{\infty} \sum_{n_1=0}^{\infty} \cdots \exp\left(-\beta \sum_i \epsilon_i n_i\right)$$

(26)

$$= \prod_i \sum_{n_i=0}^{\infty} \exp(-\beta \epsilon_i n_i)$$

(27)

for noninteracting bosons in a trap of energy levels $\{\epsilon_i\}$. Note that this partition function is different from the canonical partition function in that there is no constraint on the number of particles (i.e., the constraint $\sum_i n_i = N$ does not exist anymore) and from the grand canonical partition function in that the term $\beta \mu \mathcal{N}$ is absent inside the exponent.

In the case in which the lowest energy level is zero or negative, at least one of the sums in Eq. (27) is divergent. Such divergence does not occur in $\mathcal{Z}_C$, due to the constraint $\sum_i n_i = N$. Neither does it occur in $\mathcal{Z}_{GC}$, as the term $\beta \mu \mathcal{N}$ in the exponent "cures" the divergence.

If the lowest energy level is positive, we have

$$\mathcal{Z}^b = \prod_i \frac{1}{1 - e^{-\beta \epsilon_i}},$$

(28)

and the average occupation number of the ith level is

$$\overline{n_i} = -\frac{\partial}{\partial(\beta \epsilon_i)} \log_e \mathcal{Z}^b = \frac{1}{e^{\beta \epsilon_i} - 1},$$

(29)

so that the total average number of particles is

$$N = \sum_i \overline{n_i} = \sum_i \frac{1}{e^{\beta \epsilon_i} - 1},$$

(30)

while the total average energy is

$$\sum_i \overline{n_i} \epsilon_i = \frac{\epsilon_i}{e^{\beta \epsilon_i} - 1}.$$

(31)

Since all the energy levels are positive, these sums are convergent, and so we are not able to approach the "thermodynamical limit" ($N \to \infty$). The thermodynamical limit, i.e., the limiting case of an infinite number of particles, can be reached only in the trivial case when $T \to \infty$. This, for example, is the case for photons, in the form of modes of the electromagnetic field in a closed (finite) cavity, in which case the zero frequency mode is not populated. With increasing cavity size, one may approach as close as possible to the zero frequency mode, but then the density matrix describing the system ceases, in the limit, to be trace-class.

This problem does not arise in the case of the canonical state, as in that case, if the lowest energy level is nonzero (positive or negative), we can define $\epsilon_i^0 = \epsilon_i - \epsilon_0$ (ϵ_0 is the lowest energy level), so that we can rewrite the canonical partition function as

$$\mathcal{Z}_C^b = \exp(-\beta \epsilon_0 N) \underbrace{\sum_{n_0=0,1,\ldots} \sum_{n_1=0,1,\ldots} \cdots \exp\left(-\beta \sum_i \epsilon_i n_i\right)}_{\Sigma_i n_i = N},$$

$$= \exp(-\beta \epsilon_0 N) \mathcal{Z}_{C_0}^b,$$

(32)

where $\mathcal{Z}_{C_0}^b$ is the canonical partition function of a system of N noninteracting bosons in a trap with energy levels $\{\epsilon_i^0\}$, and with the lowest energy level vanishing. The factor $\exp(-\beta \epsilon_0 N)$ in Eq. (32) is a constant for given particle number and energy. For the case of the grand canonical state, if the lowest energy is nonzero, we again rewrite the grand canonical partition function as

$$\mathcal{Z}_{GC}^b = \prod_i \sum_{n_i=0}^{\infty} \exp[-\beta(\epsilon_i^0 - \mu_0^b) n_i] = \mathcal{Z}_{GC_0}^b,$$

(33)

where $\mathcal{Z}_{GC_0}^b$ is the grand canonical partition function of a system of N noninteracting bosons in a trap with energy levels $\{\epsilon_i^0\}$, and with the lowest energy level vanishing, and with a correspondingly changed chemical potential μ_0^b.

The discussion in this subsection so far concerned the case of noninteracting bosons. In the case of noninteracting fermions, the Pauli exclusion principle forces the n_i's in Eq. (27) to run from 0 to 1, so that the partition function in this case is (we consider "spinless," i.e., polarized fermions)

$$\mathcal{Z}^f = \prod_i \sum_{n_i=0}^{1} \exp(-\beta\epsilon_i n_i) = \prod_i [1 + \exp(-\beta\epsilon_i)]. \quad (34)$$

The average occupation numbers in this case are (we are using the same notation for bosons and fermions)

$$\overline{n}_i = -\frac{\partial}{\partial(\beta\epsilon_i)} \log_e \mathcal{Z}^f = \frac{1}{e^{\beta\epsilon_i} + 1}, \quad (35)$$

so that the total number of particles and the total energy have fixed values (for a given temperature), irrespective of the value of the lowest energy level. Therefore, again we are not able to approach the thermodynamical limit, unless $T \to \infty$.

VI. CAPACITIES OF QUANTUM CHANNELS WITH CONSTRAINED INPUTS

We now consider the case of quantum channels with constrained inputs. The motivation for such consideration is the fact that in the noiseless case, the capacities (both classical and quantum) are in some cases infinite, and thus calls for putting some realistic constraints on the setup.

Surely, the constraints will depend on the specific realization of the physical system under consideration. In anticipation of the cases that we will consider below, we put constraints on the average energy and the average number of particles of the input states. Suppose, therefore, that we have a system of particles that are sent through a channel $\mathcal{R}$, and that is governed by the Hamiltonian $\mathbb{H}$. Let the corresponding total particle number operator be $\mathbb{N}$. We only allow those inputs, ρ, to the channel that satisfies the following two constraints:

$$\text{tr}(\mathbb{H}\rho) = E \quad (36)$$

and

$$\text{tr}(\mathbb{N}\rho) = N \quad (37)$$

for certain chosen values of E and N.

Consequently, the classical capacities $\mathcal{C}$ and $\mathcal{C}^\infty$, are now respectively the same as the right-hand sides of Eqs. (5) and (6), *but* where the maximizations are over those ensembles whose average ensemble states satisfy the above constraints. For the case of $\mathcal{C}$, the input ensembles must satisfy exactly the constraints in Eqs. (36) and (36). For the case of $\mathcal{C}^\infty$, the constraints must be suitably modified. In this case, they are (cf. Ref. [27])

$$\text{tr}\left(\sum_{i=1}^{n} \mathbb{H}_i \rho\right) = nE \quad (38)$$

and

$$\text{tr}\left(\sum_{i=1}^{n} \mathbb{N}_i \rho\right) = nN, \quad (39)$$

where $\mathbb{H}_i = I \otimes I \otimes \cdots \mathbb{H} \cdots \otimes I$, i.e., it is the identity operator I at all the n sites, except at the ith one, where it is $\mathbb{H}$. Similarly, $\mathbb{N}_i = I \otimes I \otimes \cdots \mathbb{N} \cdots \otimes I$.

It seems plausible that [cf. Eq. (9)]

$$\mathcal{C}^{\infty,1} = \max\left[S(\mathcal{R}(\overline{\rho})) - \sum_i p_i S(\mathcal{R}(\rho_i)) \right], \quad (40)$$

where the maximization is over all ensembles $\{p_i, \rho_i\}$ such that the average ensemble state $\overline{\rho}$ satisfies Eqs. (36) and (37). The case of a noiseless channel with constrained inputs was considered in Ref. [27]. (Note that we are using the same notations for the capacities as those in the unconstrained case.) In the succeeding sections, we will consider the right-hand side of Eq. (40) to be the "classical capacity" of the noiseless quantum channel where the inputs are constrained by either or both the energy and number constraints.

Similarly, it seems to be true that the quantum capacities (without, or with forward, classical communication) of a quantum channel with inputs constrained by Eqs. (5) and (6), are given by the Eqs. (12) and (13), respectively, but where the maximization in the right-hand side of Eq. (12) is over those inputs that satisfies Eqs. (5) and (6). It is indeed true in the noiseless channel case, as we will show in Sec. VIII A. (Actually, in the noiseless case, all the quantum capacities defined in Sec. IV are equal.)

VII. COMMUNICATION CHANNELS THAT CARRY PHOTONS

As we have already discussed in Secs. III and IV, the classical and quantum capacities of a noiseless quantum channel that are constrained only by the dimension of the transferred state predict infinite capacities. To avoid such infinite capacities, in the case of communication channels that carry photons, one usually applies the energy constraint on the allowed states (or ensembles) that are fed into the quantum channel [5,8,9]. Suppose, therefore, that we have a system of photons that is being sent through a channel, and is described by the Hamiltonian $\mathcal{H}_p = \Sigma_i \hbar\omega_i a_i^\dagger a_i$. Here, we consider the case of classical capacity. From the considerations for massive bosons considered in Sec. VIII A, it will be apparent that all the quantum capacties defined in Sec. IV are equal to the classical capacity, in this case of photons also.

The energy constraint implies that for any ensemble, $\{p_i, \rho_i\}$ used in a classical information transfer must satisfy

$$\text{tr}(\mathcal{H}_p \overline{\rho}) = E, \quad (41)$$

where E is a preassigned value of average energy for the system, and where $\overline{\rho} = \Sigma_i p_i \rho_i$ is the average ensemble state. The classical capacity in this case is therefore obtained by maximizing the Holevo quantity $\chi(\{p_i, \rho_i\})$ over all possible ensembles $\{p_i, \rho_i\}$ that satisfy the average energy constraint in Eq. (41),

$$\mathcal{C}_E^p = \max_{\{p_i, \rho_i\} \text{ satisfying Eq. (41)}} \chi(\{p_i, \rho_i\}). \quad (42)$$

Therefore,

$$\mathcal{C}_E^p \leq \max_{\{p_i, \rho_i\} \text{ satisfying Eq. (41)}} S(\overline{\rho}). \quad (43)$$

However, the maximization on the rhs of the inequality (43) is the same as has already been considered in Sec. V D, since the number of photons is not conserved. The ensemble

that maximizes the rhs of Eq. (43) consists of the number states (Fock states), which are mutually orthogonal. Therefore, the capacity $\mathcal{C}_E^p$ is attained for this ensemble,

$$\mathcal{C}_E^p = H\left(\left\{\frac{1}{\mathcal{Z}_b}\exp\left(-\sum_i \beta\hbar\omega_i n_i\right)\right\}\right). \quad (44)$$

Here $\mathcal{Z}^b$ is given by Eq. (28), and $H(\{r_i\})=-\Sigma_i r_i \log_2 r_i$ is the Shannon entropy of a set of probabilities $\{r_i\}$. Rewriting the capacity in terms of average occupation numbers, we obtain the classical capacity of channels carrying photons as

$$\mathcal{C}_E^p = -\sum_i [\overline{n_i}\log_2 \overline{n_i} - (1+\overline{n_i})\log_2(1+\overline{n_i})], \quad (45)$$

where $\overline{n_i}$ is given by the Eq. (29). Although we have obtained this capacity by maximizing the Holevo quantity, the same result is also obtained by maximizing the accessible information in the single-copy (nonasymptotic) case [see Eq. (5)].

The above classical capacity is obtained by using the average energy constraint. Several other types of constraints on the energy are possible and all of them give the same classical capacity to lowest order (see Ref. [8]).

VIII. CHANNELS THAT CARRY MASSIVE INDISTINGUISHABLE PARTICLES

Let us now move on to the case of a communication channel that carries massive indistinguishable particles [13]. In this case, it is natural to impose the average particle number constraint along with the average energy constraint. The classical capacity in this case is the maximum of the Holevo quantity over all ensembles that satisfy both these constraints, while the quantum ones are equal to the maximum of the entropy of the state that satifies both these constraints. As we will see, both classical and quantum capacities are equal for the noiseless channel.

A system of noninteracting massive bosons exhibits a Bose Einstein condensation: Below a certain critical temperature, the number of particles in the lowest energy level (ground state) becomes a significant fraction of what is present in the whole system. In Ref. [13], we considered the channel capacity in Eq. (50) below, and showed that it can exhibit the onset of the condensation. Here we present more details of the calculations presented there. The figures presented in Ref. [13] were corresponding to the case where the bosons were confined to a three-dimensional cubic box with periodic boundary conditions. Here we consider the harmonic trap, which is actually more close to usual experimental procedures. Unless mentioned otherwise, we consider in this paper a harmonic trap in three dimensions.

Let us note that noninteracting photons, despite being bosons, do not show condensation, due to particle number nonconservation. However, effectively interacting photons may acquire an effective mass, and show condensation (see, for instance, Ref. [28]).

A. Noninteracting bosons

Suppose that a quantum channel transmits spinless noninteracting bosons described by the Hamiltonian $\mathcal{H}_b=\Sigma_i\epsilon_i a_i^\dagger a_i$,

and that they are confined to a trap of energy levels $\{\epsilon_i\}$. We will begin by considering the classical capacity.

1. Classical capacity

For a system whose average energy and average particle number are given by E and N, the allowed ensembles $\{p_i,\rho_i\}$ must satisfy the average energy constraint,

$$\mathrm{tr}(\mathcal{H}_b\bar{\rho}) = E, \quad (46)$$

and the average particle number constraint,

$$\mathrm{tr}(\mathcal{N}\bar{\rho}) = N, \quad (47)$$

where $\mathcal{N}=\Sigma_i a_i^\dagger a_i$. The classical capacity given by

$$\mathcal{C}_{E,N}^{be} = \max_{\{p_i,\rho_i\}\text{ satisfying Eqs. (46) and (47)}} \chi(\{p_i,\rho_i\}). \quad (48)$$

Therefore,

$$\mathcal{C}_{E,N}^{be} \leq \max_{\{p_i,\rho_i\}\text{ satisfying Eqs. (46) and (47)}} S(\bar{\rho}). \quad (49)$$

Once again, the maximization of the rhs of inequality (49) was considered before, in Sec. V C. There, it was discussed that the maximization of the rhs is attained by the grand canonical ensemble. The grand canonical ensemble consists of number states, which are mutually orthogonal, and therefore, the channel capacity is attained for that ensemble, so that

$$\mathcal{C}_{E,N}^{be} = H\left(\left\{\frac{1}{\mathcal{Z}_{GC}^b}\exp\left(-\beta\sum_i (\epsilon_i - \mu^b)n_i^b\right)\right\}\right). \quad (50)$$

One may rewrite the capacity in terms of the average occupation numbers as [compare with Eq. (45)]

$$\mathcal{C}_{E,N}^{be} = -\sum_i [\overline{n_i^b}\log_2 \overline{n_i^b} - (1+\overline{n_i^b})\log_2(1+\overline{n_i^b})]. \quad (51)$$

Again, the classical capacity obtained here by maximizing the Holevo quantity is the same as that obtained by maximizing the accessible information in the single-copy (nonasymptotic) case [see Eq. (5)].

2. Quantum capacity

We now consider the case of quantum capacity. Consider the noiseless quantum channel that transmits arbitrary pure states of $\mathcal{H}^{\otimes n}$, such that the energy and number of particles (and *not* their average values) in any such pure state is nE and nN, respectively. (Mixtures of such pure states are also noiselessly transmitted, by linearity.) For a noiseless channel, the average fidelity condition is automatically satisfied [see Eq. (11)]. Consequently, the corresponding quantum capacity, Q_n, irrespective of whether there is any classical side channel (see Sec. IV), is given by

$$Q_n = \log_2 \Omega(nE,nN),$$

where $\Omega(nE,nN)$ is the dimension of the subspace spanned by pure states of $\mathcal{H}^{\otimes n}$ having energy nE and number of particles nN. So, we are now in the microcanonical ensemble. (To be more precise, one should allow for small fluctuations

around nE and nN, but the derivation is similar.)

The system (described on the Hilbert space $\mathcal{H}^{\otimes n}$) consists of n identical parts, each being described on the Hilbert space $\mathcal{H}$. Since the parts are identical, each part, *on average*, has energy E and number of particles N. Consider one such part, and let $l(\epsilon, m)$ be the dimension of the subspace of that part whose energy is ϵ, and which consists of m particles. (Again, we are disregarding fluctuations.) Then,

$$\log_e \Omega(nE, nN) = \log_e \sum_{\epsilon, m} l(\epsilon, m) \Omega(nE - \epsilon, nN - m)$$

$$= \log_e \sum_{\epsilon, m} l(\epsilon, m) \exp[\log_e \Omega((n-1)E$$

$$+ E - \epsilon, (n-1)N + N - m)],$$

where we have

$$\left| \frac{E - \epsilon}{(n-1)E} \right| \ll 1, \quad \left| \frac{N - m}{(n-1)N} \right| \ll 1,$$

as we suppose that n is large (compare with e.g., [29]). Therefore, we can expand $\log_e \Omega((n-1)E + E - \epsilon, (n-1)N + N - m)$ as

$$\log_e \Omega((n-1)E + E - \epsilon, (n-1)N + N - m)$$

$$= \log_e \Omega((n-1)E, (n-1)N) + \left. \frac{\partial \log_e \Omega}{\partial E} \right|_{E=(n-1)E, N=(n-1)N}$$

$$\times (E - \epsilon) + \left. \frac{\partial \log_e \Omega}{\partial N} \right|_{E=(n-1)E, N=(n-1)N} (N - m) + \cdots .$$

$$(52)$$

Therefore,

$$\log_e \Omega(nE, nN) = \log_e \Omega((n-1)E, (n-1)N)$$

$$+ \log_e \sum_{\epsilon, m} l(\epsilon, m) e^{-\beta \epsilon + \mu^b \beta m} + \beta E - \mu^b \beta N$$

$$= \log_e \Omega((n-1)E, (n-1)N) + \log_e \mathcal{Z}_{GC}^b(E, N)$$

$$+ \beta E - \mu^b \beta N,$$

where $\mathcal{Z}_{GC}^b(E, N)$ is the grand canonical partition function of the system described on $\mathcal{H}$, and having average energy E and average particle number N. We will ultimately be interested in dividing the logarithm of $\Omega(nE, nN)$ by n and consider the limit as $n \to \infty$, and in that case, the further terms on the right-hand side of Eq. (52), after the third term, will not contribute. We now use the "fundamental equation for open systems,"

$$-k_B T \log_e \mathcal{Z}_{GC}^b = E - k_B T(\log_e 2) S(E, N) - \mu^b N,$$

where $S(E, N)$ is the von Neumann entropy of the grand canonical state of the system described on $\mathcal{H}$, and having average energy E and average particle number N. [Note that we are using the information theoretic definition of entropy as in Eq. (4), in which the multiplicative constant k_B is absent, and the logarithm is to the base 2.] Finally therefore,

$$\log_2 \Omega(nE, nN) = \log_2 \Omega((n-1)E, (n-1)N) + S(E, N).$$

Similarly,

$$\log_2 \Omega((n-1)E, (n-1)N)$$

$$= \log_2 \Omega((n-2)E, (n-2)N)$$

$$+ S(E, N),$$

so that

$$\log_2 \Omega(nE, nN) = \log_2 \Omega((n-2)E, (n-2)N) + 2S(E, N).$$

This recursion can be carried on for p times, if $n - p$ is large [30], so that we have

$$\log_2 \Omega(nE, nN) = \log_2 \Omega((n-p)E, (n-p)N) + pS(E, N).$$

$$(53)$$

We moreover demand that $\lim_{n \to \infty} \frac{p}{n} = 1$, which can be satisfied even if $n - p$ is large [31]. Since $n - p \ll n$ [32], we have

$$\lim_{n \to \infty} \frac{1}{n} \log_2 \Omega((n-p)E, (n-p)N) = 0,$$

so that

$$\lim_{n \to \infty} \frac{\log_2 \Omega(nE, nN)}{n} = S(E, N).$$

Therefore, the quantum capacity, $Q_{E,N}^{be}$, irrespective of whether we have an additional side channel for classical-information transfer, of a noiseless quantum channel that transmits arbitrary states, ρ, of $\mathcal{H}$, such that the average energy $\text{tr}(\mathcal{H}_b \rho) = E$ and the average number of particles $\text{tr}(\mathcal{N}\rho) = N$, is given by

$$Q_{E,N}^{be} = \lim_{n \to \infty} \frac{\log_2 \Omega(nE, nN)}{n} = S(E, N).$$

We therefore see that the quantum capacity is equal to the classical capacity obtained in Sec. VIII A 1.

3. Spinless bosons in a harmonic trap

Let us now consider the case of noninteracting massive spinless bosons confined in a harmonic trap. The energy levels are, therefore,

$$\hbar(\omega_x n_x + \omega_y n_y + \omega_z n_z), \quad n_x, n_y, n_z = 0, 1, 2, \cdots . \quad (54)$$

In evaluating the capacity numerically, we must cut off the infinite sequence of energy levels at some point. The evaluated "capacity" of course depends on the point of the cutoff. In Fig. 1, we plot the channel capacity for the case of 10^4 bosons (with respect to temperature), where we show the "capacities" for the different points of the cutoff. In this case, the convergence is obtained roughly at about $n_x = n_y = n_z = 300$. We see that the curves show a sort of fracture at a certain temperature, changing their behavior from being concave to being convex with respect to temperature. This can be seen more clearly in Fig. 2, where we plot the derivative of the plots in the preceding figure. For calculating the derivative, we use a four-point formula,
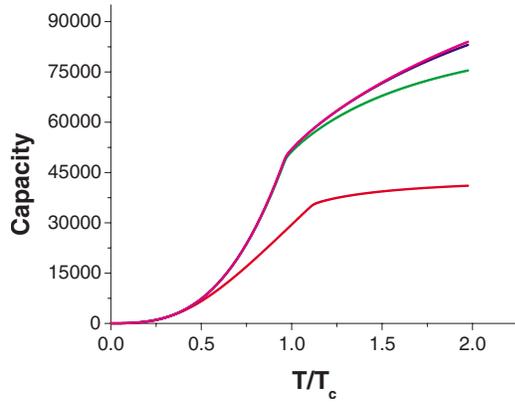
FIG. 1. (Color online) Capacity (in bits or qubits) as a function of the dimensionless variable T/T_c for $N=10^4$ massive bosons in a harmonic trap, with different summation boundaries. T_c denotes the critical temperature in the thermodynamic limit. Therefore, $T_c = \frac{\hbar\omega_{ho}}{k_B}\left(\frac{N}{\zeta(3)}\right)^{1/3}$, where $\omega_{ho} = (\omega_x\omega_y\omega_z)^{1/3}$ is the geometrical average of the frequencies and ζ is the Riemann zeta function. For the purpose of all the figures in this paper, we consider a harmonic trap, and the case when $\omega_x = \omega_y = \omega_z$. The four curves (from bottom to top) represent cutoffs at $n_x = n_y = n_z = 40$, 120, 220, and 300, respectively. Note that the last two curves almost coincide.

$$\frac{df}{dx} = \frac{1}{12\Delta x}[f(x-2\Delta x) - 8f(x-\Delta x) + 8f(x+\Delta x)$$

$$- f(x+2\Delta x)] + O(\Delta x^4). \tag{55}$$

In the thermodynamic limit, this bending causes the derivative to have a discontinuity, and the corresponding temperature is the critical temperature of the condensation. Figure 3 shows the development of this discontinuity in the derivative of the capacity, with increasing average total particle number N. The corresponding capacities are plotted in Fig. 4. As seen in Figs. 3 and 4, the critical temperature corresponding to the

FIG. 3. (Color online) Derivative of the capacity (in bits or qubits) as function of the dimensionless variable T/T_c for different number of bosons. Curves (from bottom to top) represent 100, 500, 1000, 5000, and 10000 bosons, respectively.

fracture grows with N, as expected. The fractures indicate the onset of the condensation for the corresponding values of N. The gap between the temperature corresponding to the fracture and the thermodynamical critical temperature is known to exist, and estimated values of the gap has also been given (see Ref. [33]). Also, the capacities grow with the number of particles, as expected.

It is known that the dimension of the system under consideration plays a role in determining whether a condensation exists. For example, in the case of harmonic traps, the 3D and 2D traps exhibits condensations, while the 1D case does not show a condensation (see e.g., [33]). The cases considered in the previous (and latter) figures are 3D harmonic traps. In Fig. 5, we compare the qualitative behavior of the derivatives of the capacities for 2D and 1D.

Going back to the case of the 3D harmonic trap, we consider energy as a function of the temperature, and see that the criticality is clearly visible in these curves. In Fig. 6, we plot the energy and its derivative with respect to temperature, for
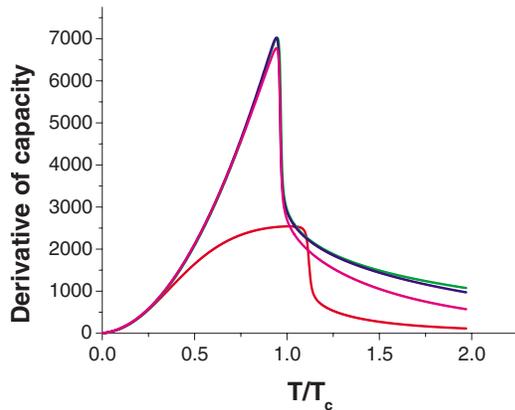

FIG. 2. (Color online) Derivative of the capacity (in bits or qubits) as function of the dimensionless variable T/T_c for $N=10^4$ bosons. Note that the number of bosons is the same as that in the previous figure. Moreover, the curves in this figure (from bottom to top) represent the cutoffs at 40, 120, 220, and 300, respectively, which are exactly the same as those in the previous figure. Note that again the last two curves almost coincide.
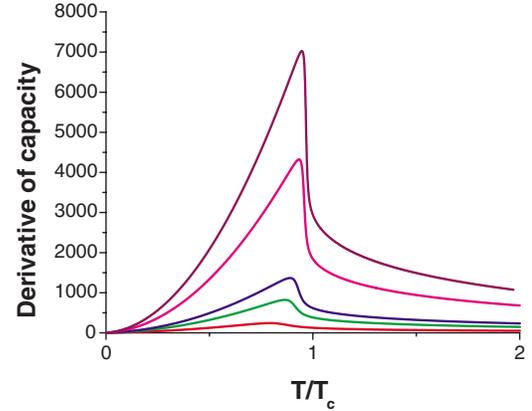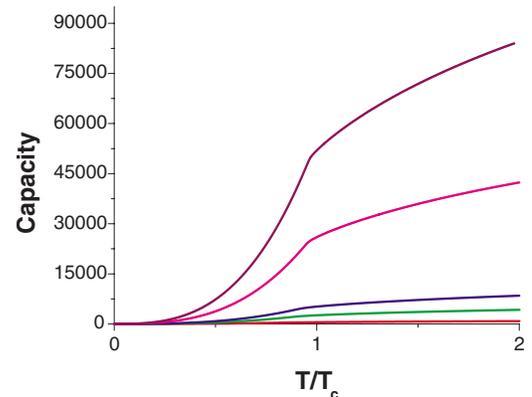

FIG. 4. (Color online) Capacity (in bits or qubits) as a function of the dimensionless variable T/T_c for different number of bosons. At the critical temperature, the capacity changes its behavior from being concave to being convex (with respect to temperature). Again, the curves (from bottom to top) represent 100, 500, 1000, 5000, and 10000 bosons, respectively.
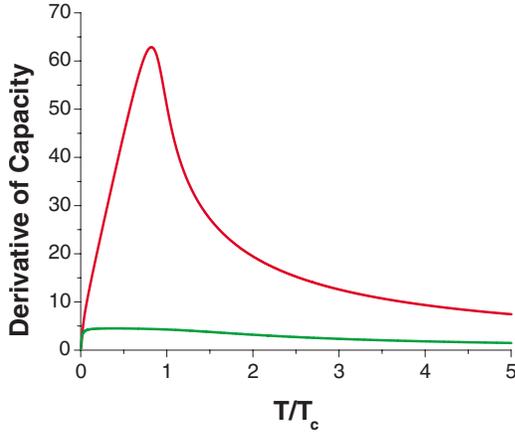
FIG. 5. (Color online) Derivatives of the capacities (in bits or qubits) are plotted as functions of the temperature for $N=100$ bosons in one- and two-dimensional traps. The upper curve represents the 2D case, where the horizontal axis is of the dimensionless variable T/T_c^{2D}. And $T_c^{2D}=\frac{\hbar\omega_{ho}^{2D}}{k_B}\left(\frac{N}{\zeta(2)}\right)^{1/2}$, where $\omega_{ho}^{2D}=(\omega_x\omega_y)^{1/2}$ [33]. For the figure, we consider the case in which $\omega_x=\omega_y$. In the one-dimensional case (lower curve), the horizontal axis is of the dimensionless variable T/T_c^{1D}, where $T_c^{1D}=\frac{\hbar\omega_{ho}}{k_B}\frac{N}{\log_e(2N)}$ [34]. It is clear from the figure that the capacity of a one-dimensional trap does not show any onset of condensation, while that for a two-dimensional trap clearly does show it.

500 spinless bosons in a harmonic trap. Note that the capacity plotted with respect to energy does not indicate the condensation.

B. Noninteracting fermions in a harmonic trap

Let us now move over to the case of noninteracting fermions. Similar arguments as in the preceeding subsection imply that the capacities of N spin-s noninteracting fermions in a trap with energy levels $\{\epsilon_i\}$ are given by
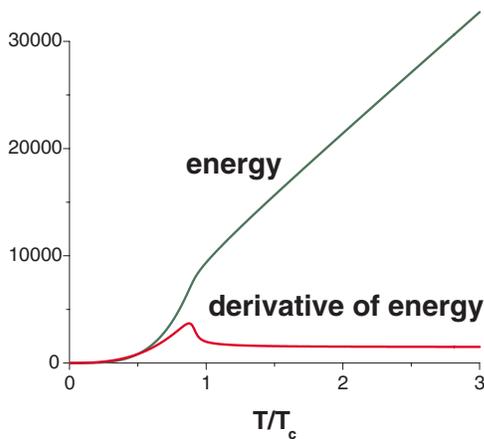

FIG. 6. (Color online) For $N=500$ bosons, the dimensionless variable energy/$\hbar\omega$ (where $\omega_x=\omega_y=\omega_z=\omega$) and its derivative, are plotted against the dimensionless variable T/T_c. Energy behaves quite similarly as the capacity, in that it changes its behavior from being convex to being concave, with respect to temperature, at the criticality. This is more clear from the derivative of the energy.
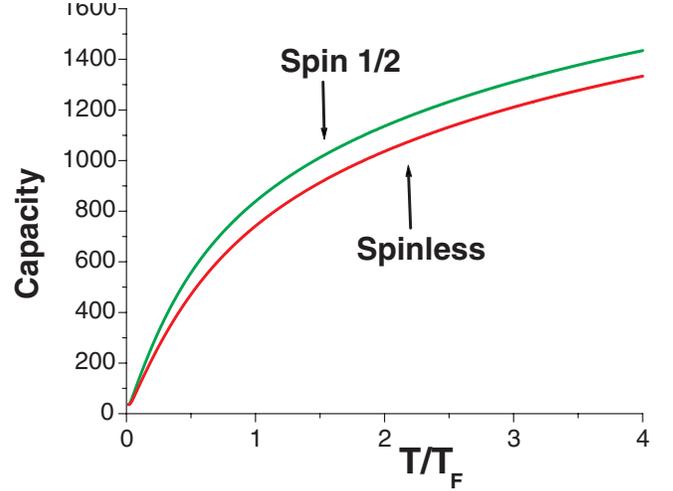

FIG. 7. (Color online) For $N=100$ fermions, the capacity (in bits or qubits) of spin-1/2 fermions is compared with that of spinless fermions. The horizontal axis is of the dimensionless variable T/T_F, where T_F is the Fermi temperature for spin-1/2 fermions. Note that we use the same Fermi temperature for *both* the curves. The Fermi temperature for a harmonic trap is given by $T_F=(\hbar\omega_{ho}/k_B)\left(\frac{6N}{g}\right)^{1/3}$.

$$\mathcal{C}_{E,N}^{fd}=Q_{E,N}^{fd}$$
$$=H\left(\left\{\frac{1}{\mathcal{Z}_{GC}^f}\exp\left(-\beta\sum_i\sum_{mag=-s}^s(\epsilon_i-\mu^f)n_{i,mag}^f\right)\right\}\right),$$
$$(56)$$

where

$$\mathcal{Z}_{GC}^f=\prod_i\{1+\exp[-\beta(\epsilon_i-\mu^f)]\}^g,\qquad(57)$$

the power $g=2s+1$ being present due to the degeneracy of the spin states. The suffix "mag" represents the magnetic quantum number. μ^f represents the chemical potential in this case. The channel capacity in this case is reached by the fermionic grand canonical ensemble,

$$\left\{\frac{1}{\mathcal{Z}_{GC}^f}\exp\left(-\beta\sum_i\sum_{mag=-s}^s(\epsilon_i-\mu^f)n_{i,mag}^f\right)\right.,$$

$$\left.\left|n_{0,-s}^f,n_{0,-s+1}^f,\ldots,n_{0,s}^f,n_{1,-s}^f,n_{1,-s+1}^f,\ldots,n_{1,s}^f,\ldots\right\rangle\right\},\quad(58)$$

where the elements of the ensemble runs over all combinations of the $n_{i,mag}^f$'s ($n_{i,mag}^f=0,1$, and mag$=-s,-s+1,\ldots,s$). Again the capacities can be rewritten in terms of the average occupation numbers,

$$\overline{n_i^f}=\frac{g}{e^{\beta(\epsilon_i-\mu^f)}+1},\qquad(59)$$

as

$$\mathcal{C}_{E,N}^{fd}=Q_{E,N}^{fd}=-g\sum_i\left[\frac{\overline{n_i^b}}{g}\log_2\frac{\overline{n_i^b}}{g}-\left(1+\frac{\overline{n_i^b}}{g}\right)\log_2\left(1+\frac{\overline{n_i^b}}{g}\right)\right].$$
$$(60)$$

In Fig. 7, we compare the capacities of spinless fermions with that of spin 1/2 fermions in a harmonic trap.

Noninteracting fermions do not exhibit a condensation. However, interacting fermions can exhibit Cooper pairing, and consequently superfluid BCS transition. The channel capacity in such case also indicates the onset of the BCS transition (see Ref. [13]).

IX. FERMIONS ARE BETTER CARRIERS OF INFORMATION THAN BOSONS

Numerical simulation with several values of the total number of particles, with several types of traps, and over a large range of temperature reveals that fermions are better carriers of information than bosons. For sufficiently high temperatures, we have been able to obtain analytical results in this respect, as we have already presented in the following theorem from Ref. [13]. Here, we give a detailed proof of the theorem. We have already proven that the classical and quantum capacities are the same in the case under study. For definiteness, we will consider only the classical capacity in this section and the succeeding one.

Theorem. For power-law potential traps (with power γ and dimension d), and for sufficiently high temperatures, the capacity of spinless fermionic channel is better than that of spinless bosonic channel when

$$\frac{1}{\gamma} + \frac{1}{2} > \frac{1}{d}. \tag{61}$$

Note that a power-law potential with power γ and dimension d is given by $V(x_1, x_2, \ldots, x_d) = r^\gamma$, where $r = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}$, in the d-dimensional Cartesian space $(x_1, x_2, \ldots, x_d)$.

Proof. Let us start with the case of (spinless) bosons and perform the high-temperature expansion. First we expand the fugacity

$$z_b = \exp(\beta \mu^b) \tag{62}$$

in powers of N/S_1, where

$$S_k = \sum_i \exp(-k\beta \epsilon_i), \quad k = 1, 2, \ldots . \tag{63}$$

Note that

$$
\begin{aligned}
N &= \sum_i \overline{n_i^b} \\
&= \sum_i \frac{1}{\frac{1}{z_b} e^{\beta \epsilon_i} - 1} = \sum_i z_b \exp(-\beta \epsilon_i)[1 - z_b \exp(-\beta \epsilon_i)]^{-1} \\
&= z_b S_1 + z_b^2 S_2 + z_b^3 S_3 + \cdots .
\end{aligned} \tag{64}
$$

Suppose now that

$$z_b = b_0 + b_1 \frac{N}{S_1} + b_2 \left(\frac{N}{S_1}\right)^2 + b_3 \left(\frac{N}{S_1}\right)^3 + \cdots . \tag{65}$$

Substituting z_b, as in Eq. (65), into Eq. (64), and comparing different powers of N/S_1, we find that

$$b_0 = 0, \quad b_1 = 1,$$

$$b_2 = -\frac{S_2}{S_1}, \quad b_3 = \frac{2S_2^2 - S_1 S_3}{S_1^2}. \tag{66}$$

We are now in a position to expand $\mathcal{C}_{E,N}^{be}/\log_2 e$ in powers of N/S_1. We use the expression in Eq. (51) for this purpose. In Eq. (51), the substitutions in the different terms are done as follows. For the term $\overline{n_i^b} \log_e \overline{n_i^b}$, for $\overline{n_i^b}$, we substitute

$$\overline{n_i^b} = z_b \exp(-\beta \epsilon_i)[1 - z_b \exp(-\beta \epsilon_i)]^{-1} \tag{67}$$

$$= z_b e^{-\beta \epsilon_i} + z_b^2 e^{-2\beta \epsilon_i} + z_b^3 e^{-3\beta \epsilon_i} + \cdots \tag{68}$$

and for $\log_2 \overline{n_i^b}$, we use Eq. (67) to write

$$
\begin{aligned}
\log_e \overline{n_i^b} &= \log_e z_b - \beta \epsilon_i - \log_e [1 - z_b \exp(-\beta \epsilon_i)] \\
&= \log_e z_b - \beta \epsilon_i + \left(z_b e^{-\beta \epsilon_i} + \frac{1}{2} z_b^2 e^{-2\beta \epsilon_i} + \frac{1}{3} z_b^3 e^{-3\beta \epsilon_i} \right. \\
&\quad \left. + \cdots \right).
\end{aligned} \tag{69}
$$

and this is then used for the substitution. For the term $(1 + \overline{n_i^b}) \log_e(1 + \overline{n_i^b})$, we use Eq. (68) to substitute for $\overline{n_i^b}$ in both places, and then $\log_e(1 + \overline{n_i^b})$ is expanded in powers of $\overline{n_i^b}$.

A similar calculation is done for spinless fermions. In this case, the fermionic fugacity

$$z_f = \exp(\beta \mu^f) \tag{70}$$

can be expanded as

$$z_f = f_0 + f_1 \frac{N}{S_1} + f_2 \left(\frac{N}{S_1}\right)^2 + f_3 \left(\frac{N}{S_1}\right)^3 + \cdots, \tag{71}$$

in which the f_i's are obtained from the average particle number conservation equation,

$$
\begin{aligned}
N &= \sum_i \overline{n_i^f} = \sum_i z_f \exp(-\beta \epsilon_i)[1 + z_f \exp(-\beta \epsilon_i)]^{-1} \\
&= z_f S_1 - z_f^2 S_2 + z_f^3 S_3 - \cdots
\end{aligned} \tag{72}
$$

as

$$f_0 = 0, \quad f_1 = 1,$$

$$f_2 = \frac{S_2}{S_1}, \quad f_3 = \frac{2S_2^2 - S_1 S_3}{S_1^2}. \tag{73}$$

Note that these are the same as in the case of bosons, except for the sign in f_2.

We perform the calculation up to the third order, and find that

$$\frac{\mathcal{C}_{E,N}^{be}}{\log_2 e} = \sum_{i=1}^{3} \alpha_i^b \left(\frac{N}{S_1}\right)^i + \beta_1^b \frac{N}{S_1} \log_e \frac{N}{S_1} + \beta_2^b \left(\frac{N}{S_1}\right)^2 \log_e \frac{N}{S_1} \tag{74}$$

plus higher-order terms, whereas

$$\frac{\mathcal{C}^{fd}_{E,N}}{\log_2 e} = \sum_{i=1}^{3} \alpha_i^f \left(\frac{N}{S_1}\right)^i + \beta_1^f \frac{N}{S_1} \log_e \frac{N}{S_1} + \beta_2^f \left(\frac{N}{S_1}\right)^2 \log_e \frac{N}{S_1} \tag{75}$$

plus higher-order terms. The coefficients of first-order perturbation are equal,

$$\alpha_1^b = \alpha_1^f = S_1 + D_1, \tag{76}$$

where we have set

$$D_k = \sum_i \beta \varepsilon_i \exp(-k\beta\varepsilon_i), \quad k = 1, 2, \dots . \tag{77}$$

In the next order, they differ by a sign,

$$\alpha_2^b = -\alpha_2^f = \frac{S_2}{2} - \frac{S_2}{S_1} D_1 + D_2. \tag{78}$$

The third-order perturbation coefficients are again equal,

$$\alpha_3^b = \alpha_3^f = -3S_2 + \frac{S_3}{3} + \frac{2S_2^2}{S_1} + \frac{2S_2^2 - S_1 S_3}{S_1^2} D_1 - \frac{2S_2}{S_1} D_2 + D_3. \tag{79}$$

Also,

$$\beta_1^b = \beta_1^f = -S_1,$$

$$\beta_2^b = 0, \quad \beta_2^f = 2S_2. \tag{80}$$

Now, up to third order, the only coefficients that are different are those of $(N/S_1)^2$ and $(N/S_1)^2 \log_e(N/S_1)$. We have

$$0 = \beta_2^b < \beta_2^f. \tag{81}$$

Also

$$\alpha_2^b = S_2 \left(\frac{1}{2} - \beta \frac{D_1}{S_1} + \beta \frac{D_2}{S_2}\right). \tag{82}$$

We now evaluate $\frac{D_1}{S_1}$ and $\frac{D_2}{S_2}$ by using the density of states [35],

$$\varsigma(\epsilon) = \int d^d p \, d^d x \, \delta\left(\frac{p^2}{2m} + V(\vec{r}) - \epsilon\right) \tag{83}$$

for a system in d-dimensionsal Cartesian space $\vec{r} = (x_1, x_2, \dots, x_d)$, trapped in a potential $V(\vec{r})$. $\vec{p} = (p_1, p_2, \dots, p_d)$ denotes the respective momentum, with $p = |\vec{p}|$. The integration in Eq. (83) is over the phase space. We may rewrite $\varsigma(\epsilon)$ as [36]

$$\varsigma(\epsilon) = \text{const} \times \int dx_1 \cdots dx_d (\epsilon - V)^{(d-2)/2}. \tag{84}$$

Let us now consider the potential as [35]

$$V(\vec{r}) = r^\gamma, \tag{85}$$

where $r = \sqrt{x_1^2 + \cdots + x_d^2}$. In this case,

$$\varsigma(\epsilon) = \text{const} \times \epsilon^{d/\gamma + [(d-2)/2]} \int d\tilde{x}_1 \cdots d\tilde{x}_d$$

$$\times [1 - (\tilde{x}_1^2 + \cdots + \tilde{x}_d^2)^{\gamma/2})^{(d-2)/2},$$

so that

$$\varsigma(\epsilon) = \text{const} \times \epsilon^{d/\gamma + [(d-2)/2]}, \tag{86}$$

where $x_i = \epsilon^{\frac{1}{\gamma}} \tilde{x}_i$ $(i=1,\dots,d)$. We are now ready to calculate the sums in Eq. (82). We have

$$\frac{D_1}{S_1} = \frac{\displaystyle\int \beta\epsilon \exp(-\beta\epsilon)\varsigma(\epsilon)d\epsilon}{\displaystyle\int \exp(-\beta\epsilon)\varsigma(\epsilon)d\epsilon}$$

$$= \frac{\displaystyle\int \beta\epsilon \exp(-\beta\epsilon)\epsilon^{d/\gamma+[(d-2)/2]}d\epsilon}{\displaystyle\int \exp(-\beta\epsilon)\epsilon^{d/\gamma+[(d-2)/2]}d\epsilon} = \frac{\Gamma\left(\frac{d}{\gamma} + \frac{d-2}{2} + 1\right)}{\Gamma\left(\frac{d}{\gamma} + \frac{d-2}{2}\right)}$$

$$= \frac{d}{\gamma} + \frac{d}{2}. \tag{87}$$

The integrations over ϵ are performed from 0 to ∞. Similarly, one can calculate $\frac{D_2}{S_2}$, and it is equal to $\frac{1}{2}\left(\frac{d}{\gamma} + \frac{d}{2}\right)$. Therefore,

$$\alpha_2^b = \frac{S_2}{2}\left[1 - \left(\frac{d}{\gamma} + \frac{d}{2}\right)\right]. \tag{88}$$

Therefore,

$$\alpha_2^b \leqslant 0 \leqslant \alpha_2^f \tag{89}$$

holds when

$$\frac{1}{\gamma} + \frac{1}{2} > \frac{1}{d}. \tag{90}$$

That is, the fermionic capacity is greater than the bosonic one in such cases. However, we must now check for which potentials and dimensions, the above perturbation technique is systematic.

In the expansion of the capacity in terms of N/S_1 given in Eq. (74) for bosons and in Eq. (75) for fermions, the coefficients of $\frac{N}{S_1}$, $\frac{N}{S_1}\log_e\frac{N}{S_1}$, $\left(\frac{N}{S_1}\right)^2$, $\left(\frac{N}{S_1}\right)^2\log_e\frac{N}{S_1}$, and $\left(\frac{N}{S_1}\right)^3$ are all of the order $\beta^{-(d/\gamma+d/2)}$, since

$$S_1 = \int \exp(-\beta\epsilon)\epsilon^{-\{d/\gamma+[(d-2)/2]\}}d\epsilon \sim \beta^{-(d/\gamma+d/2)}, \tag{91}$$

and

$$D_1 = \int \beta\epsilon \exp(-\beta\epsilon)\epsilon^{-\{d/\gamma+[(d-2)/2]\}}d\epsilon \sim \beta^{-(d/\gamma+d/2)}. \tag{92}$$

Moreover, $\frac{N}{S_1} \sim \beta^{(d/\gamma+d/2)}$. For systematics of the expansion in Eq. (74) for bosons, we need that the orders of β in $\alpha_1^b \frac{N}{S_1}$, $\beta_1^b \frac{N}{S_1}\log_e\frac{N}{S_1}$, $\alpha_2^b\left(\frac{N}{S_1}\right)^2$, and $\alpha_3^b\left(\frac{N}{S_1}\right)^3$ should be in increasing order of β. This demand leads to the following condition:
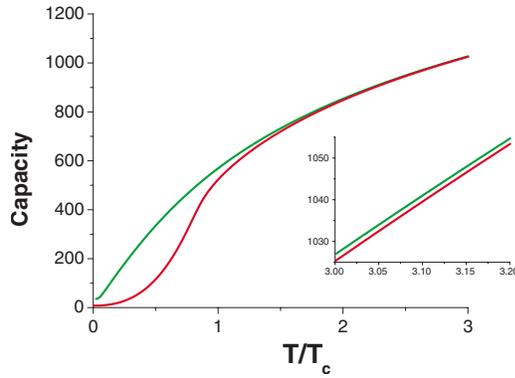
FIG. 8. (Color online) Bosons vs fermions. We plot the capacities (in bits or qubits) of spinless bosons and spinless fermions with respect to the dimensionless variable T/T_c, for 100 particles. Note that the horizontal axis represents T/T_c for *both* the curves. The lower curve is for bosons. The figure clearly agrees with our theorem. Moreover, this figure, along with other numerical simulations that we have performed, indicates that the theorem is true even for low temperatures. The inset compares the behavior in an exemplary case at high temperatures.

$$\frac{2d}{\gamma} + d > \frac{d}{\gamma} + \frac{d}{2} > 1 > 0. \tag{93}$$

This requires that $\frac{1}{\gamma} + \frac{1}{2} > \frac{1}{d}$, which is the same as the condition required for fermions having a higher capacity than bosons. Similar calculation for the fermions leads to the same requirement.

Lastly, note that N/S_1 tends to zero, which implies that $T \to \infty$. This completes the proof. □

Remark 1. The condition in Eq. (61) includes, e.g., the harmonic trap in 2D and 3D, the 3D rectangular box, and the 3D spherical box.

Remark 2. In the proof, we work up to order $(N/S_1)^3$, and so the theorem holds for quite moderate temperatures.

At this point, please note that we have numerically checked that the statement of the theorem holds also for low temperatures for the harmonic trap and for the 3D rectangular box [13]. In Fig. 8, we compare the capacities of channels carrying bosons and fermions for $N=100$ for the case of a harmonic trap.

X. LOW-TEMPERATURE BEHAVIOR

As we have already stressed, although the low-temperature behavior of the capacity is not covered by the Theorem in the preceeding section, numerical simulations seems to indicate that the statement of the Theorem is indeed true for lower temperatures. In this section, we want to point out that the statement can be proven analytically in some circumstances, for low temperatures. For example, for the case of a bounded volume V, containing N particles at

temperature T, if the particles are bosons, then the capacity at low temperatures [specifically for $T \lesssim T_c$, where in this case $T_c = \hbar^2 \frac{2\pi}{mk_B} \left(\frac{N}{V\zeta(3/2)}\right)^{2/3}$] is given by (see, e.g., [25])

$$\frac{C_{V,N,T}^{be}}{\log_2 e} = \frac{5V\zeta(5/2)}{2\hbar^3} \left(\frac{mk_B T}{2\pi}\right)^{3/2},$$

i.e., the bosonic capacity $\sim T^{3/2}$. On the other hand, if the particles are fermions, then the capacity, for $T \lesssim T_F$ [where in this case, $T_F = \frac{\hbar^2}{2mk_B}\left(\frac{6\pi^2 N}{gV}\right)^{2/3}$], is given by (see, e.g., [25])

$$\frac{C_{V,N,T}^{fd}}{\log_2 e} = \frac{\pi^2}{2}\frac{T}{T_F},$$

i.e., the fermionic capacity scales as T. Clearly, the fermionic capacity is higher than the bosonic one for sufficiently low temperatures in a bounded volume, such as a 3D box.

In the case of a harmonic trap in 3D, the bosonic capacity scales as T^3 (see, e.g., [37]), when the temperature is below critical. For sufficiently low temperatures, the fermionic capacity scales as T, which can be estimated by using the Sommerfeld expansions of Fermi functions. Therefore, the fermionic capacity wins once again.

XI. CONCLUSIONS

In this paper, we have investigated the classical as well as the quantum capacity of noiseless quantum channels, carrying massive particles. We have considered spinless noninteracting bosons and fermions. Noninteracting bosons exhibit Bose-Einstein condensation, and we have shown that this critical behavior can also be observed by studying the capacity of a quantum channel carrying bosons. We show that the capacity of such channels is concave (with respect to temperature) above the critical temperature, while it is convex (with respect to temperature) below that point. This criticality is absent in the case of fermions, as expected. We have numerically evaluated the capacities of bosons and fermions for different numbers of particles. In the case of bosons, even in the case of a small number of particles, say 100, condensation can be observed from the qualitative change in behavior of the capacity.

We have also shown analytically that a channel carrying bosons is not as good a medium for transferring classical as well as quantum information. This is true for a wide range of potentials that can be currently created in the laboratory. The analytical calculation for power-law potentials holds for moderate temperatures. However, numerically we have checked that this is true even for low temperatures. It is tempting to believe that such superiority of fermions over bosons is generic, at least for power-law potentials. In special cases, we have also considered the low-temperature behavior analytically, and have shown that for sufficiently low temperatures, the fermionic capacity is again higher than the bosonic one, for the 3D box and the 3D harmonic trap.

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[30] This is because the last relation that we use to obtain the relation (53) is $\log_2 \Omega((n-p+1)E, (n-p+1)N) = \log_2 \Omega((n-p)E, (n-p)N) + S(E, N)$, which assumes that $n-p$ is large.

[31] For example, one may choose $p = t - \lfloor \log_e t \rfloor$, and $n = t$, for any positive integer t, where $\lfloor \cdot \rfloor$ denotes the integer part of its argument.

[32] Actually, $\lim_{n \to \infty} \frac{n-p}{n} = 0$, as $\lim_{n \to \infty} \frac{p}{n} = 1$.

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[36] To arrive at Eq. (84) from Eq. (83), we transform the Cartesian momentum coordinates in the integral in Eq. (83) to hyperspherical coordinates, to have

$$\varsigma(\epsilon) = \text{const} \times \int p^{d-1} dp\, d^d x\, \delta\left(\frac{p^2}{2m} + V(\vec{r}) - \epsilon \right) = \text{const}$$

$$\times \int \varpi^{(d-2)/2} d\varpi\, d^d x\, \delta(\varpi + V(\vec{r}) - \epsilon),$$

where we have substituted $p^2/2m = \varpi$. Equation (84) follows directly from the last equation.

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