Quantification of quantum correlation of ensembles of states

Michał Horodecki,[1] Aditi Sen(De),[1,2,3] and Ujjwal Sen[1,2,3]
[1]*Instytut Fizyki Teoretycznej i Astrofizyki, Uniwersytet Gdański, PL-80-952 Gdańsk, Poland*
[2]*ICFO-Institut de Ciencies Fotoniques, Mediterranean Technology Park, 08860 Castelldefels (Barcelona), Spain*
[3]*Institut für Theoretische Physik, Universität Hannover, D-30167 Hannover, Germany*
(Received 7 July 2006; published 26 June 2007)

We present a measure of quantum correlation of an ensemble of multiparty states. It is based on the idea of minimal entropy production in a measurement whose elements are locally distinguishable. It is shown, for pure ensembles, to be a relative entropy distance from a set of ensembles. For pure bipartite ensembles, which span the whole bipartite Hilbert space, the measure is bounded below by the average entanglement. We naturally obtain a monotonicity axiom for any measure of quantum correlation of ensembles for single systems. We evaluate the measure for certain cases. Subsequently we use this measure to propose a complementarity relation between our measure and the accessible information obtainable about the ensemble under local operations. The measure is well defined even for the case of a single system, where the complementarity relation is seen to be yet another face of the "Heisenberg uncertainty relation."

DOI: 10.1103/PhysRevA.75.062329 PACS number(s): 03.67.Mn, 03.65.Ud

I. INTRODUCTION

Quantifying quantum correlation of states is a much studied problem in quantum information (see, e.g., Refs. [1–5]) Several measures have been proposed for such quantification. Prominent among them are distillable entanglement [1], entanglement of formation [1], relative entropy of entanglement [2], etc. One of the reasons for the fascination is that quantum correlations are a resource for several nonclassical tasks, like dense coding [6] and quantum teleportation [7].

However to our knowledge, the problem of quantification of quantum correlation of an ensemble of states has never been studied. Such a quantification has potential applications in quantitative studies of quantum cryptographic tasks [8–14]. In this paper we propose a measure of quantum correlation ($\mathcal{Q}$) of an ensemble of states. A trivial way to quantify quantum correlation of an ensemble of states is to take an average of an individual property of the constituent states in the ensemble. For example, one could take the average entanglement of formation [1] of the ensemble states. However, such an averaging over a property of the individual states cannot capture the complexity of the ensemble as a whole. To take into account the structure of the ensemble as a whole, the measure must depend on the ensemble as a whole. A simple measure of quantum correlation of multiparty ensembles could be the difference $\mathbb{D}$ between the globally accessible information and the locally accessible information. However, accessible information is an operationally useful quantity, and one aim of defining $\mathcal{Q}$ is to estimate accessible information under different sets of operations. Moreover, $\mathbb{D}$ vanishes for single-party ensembles, whereas $\mathcal{Q}$ will be seen to be nontrivial also in such cases.

We begin in Sec. II with a discussion of the concept of accessible information, which will be needed later on in the paper. We define the measure in Sec. III, and present the motivations behind such a definition. In Sec. IV, we relate our measure to relative entropy distances. We show that for pure ensembles, it can be seen as a relative entropy distance from a set of "classical" ensembles. For a pure bipartite ensemble, which spans the whole bipartite Hilbert space, we show that measure is bounded below by the average relative entropy of entanglement [2]. In Sec. V, we discuss the monotonicity properties of the measure, and propose the first axioms for any measure of quantumness of an ensemble of a single system. We then evaluate the measure for the case of four Bell states (given with equal prior probabilities) in Sec. VI. We also evaluate the measure for other more general ensembles. We subsequently propose (Sec. VII) a complementarity relation (see Refs. [15,16]) between this measure of quantum correlation of an ensemble and the accessible information obtainable about the ensemble, under the relevant class of allowable operations. In the case of a single system, this complementarity is proven under a certain assumption, and is seen to be yet another face of the "Heisenberg uncertainty relation" (Sec. VIII). Importantly, both the information gathering and disturbance terms in this relation are information-theoretic, in contrast to, e.g., Ref. [17]. We give our conclusions in Sec. IX.

II. ACCESSIBLE INFORMATION

In this section, we discuss the concept of accessible information, that quantifies the amount of classical information that can be decoded from an ensemble of quantum states. Suppose therefore that a source encodes a classical message x, with probability p_x, in a quantum state ϱ_x, and sends the resulting ensemble

$$\mathcal{E} = \{p_x, \varrho_x\}$$

to one or more receivers. Let X be the (classical) random variable corresponding to the message variable x. The receiver(s) are able to perform a set of quantum operations Λ. The task of the receiver(s) is to gather as much information as possible (by using the allowed operations in Λ) about the index x. To do so, the receiver(s) performs (perform) a measurement M (that is in Λ), that gives the result m, with probability q_m. Let the corresponding post-measurement ensemble be

$$\{p_{x|m}, \varrho_{x|m}\}.$$

Let Y^M be the (classical) random variable corresponding to the measurement outcomes in the measurement M. The information gathered can be quantified by the classical mutual information between the random variables X and Y^M [18]:

$$I(X{:}Y^M) = H(\{p_x\}) - \sum_m q_m H(\{p_{x|m}\}). \tag{1}$$

Here

$$H(\{r_x\}) = -\sum_x r_x \log_2 r_x$$

is the Shannon entropy of the probability distribution $\{r_x\}$. Throughout the paper, we calculate all the quantities on amounts of information transfer in bits (binary digits). Note that the mutual information can be seen as the difference between the initial disorder and the (average) final disorder. The receiver(s) are interested to obtain the maximal information, which is the maximum of $I(X{:}Y^M)$ over all measurement strategies. This quantity is called the accessible information under Λ:

$$I_{\rm acc}^{\Lambda} = \max I(X{:}Y^M), \tag{2}$$

where the maximization is over all measurement strategies in Λ. In particular, if there are more than one receivers who are allowed to act locally (with quantum mechanically allowed operations) and communicate over a classical channel, then the corresponding accessible information is called "locally accessible information" and denoted as $I_{\rm acc}^{\rm LOCC}$, where LOCC is an abbreviation for local operations and classical communication.

III. MOTIVATION AND DEFINITION OF THE MEASURE

Multiparty states are not necessarily locally distinguishable, even if they are mutually orthogonal (see, e.g., Ref. [19]). However, a geometrical concept (such as nonorthogonality, for global operations), that separates locally distinguishable ensembles from locally indistinguishable ones is absent. One of our goals is to provide such a notion of "local nonorthogonality." The idea is that locally distinguishable ensembles are to be considered as "classical" ("nonquantum"). A locally indistinguishable ensemble will of course not "match" with any locally distinguishable one. The amount of quantumness of the locally indistinguishable ensemble will then be the amount of this disharmony, quantified by the minimal entropy production when the ensemble is dephased (globally) in a measurement, whose elements are locally distinguishable. Note here that there are general methods and results to determine local distinguishability of ensembles in certain cases [20,21]. These results can be used to determine the value of the measure, which can then be used, as we show later, to provide bounds on locally accessible information.

Consider therefore an ensemble of bipartite states

$$\mathcal{E} = \{p_x, \varrho_x^{AB}\}_{x=1}^N,$$

that are not necessarily orthogonal. (If the probabilities p_i are not explicitly mentioned, they are assumed to be equal.) Let $\mathcal{A}$ be the Hilbert space spanned by the support of the states ϱ_x^{AB}. Let $\{\eta_i\}_{i=1}^n$ be a measurement of $\mathcal{H}_A \otimes \mathcal{H}_B$, such that the vector of ranks of η_i is majorized by the corresponding vector of the ensemble states [22]. Consider the part of $\{\eta_i^{AB}\}_{i=1}^n$ which has a nonzero overlap with $\mathcal{A}$ [23]. The normalized operators

$$\tilde{\eta}_i = \eta_i/\mathrm{tr}(\eta_i)$$

are states, and can be distinguishable or indistinguishable under local operations and classical communication [24]. Let $\mathcal{D}_{\mathcal{A}}$ be the set of all such measurements $\{\eta_i^{AB}\}_{i=1}^n$ whose overlap with $\mathcal{A}$, if normalized, is LOCC distinguishable.

The measure of quantum correlation $\mathcal{Q}$ for the ensemble $\mathcal{E}$ is defined as the minimal entropy produced when dephased (measured) by a measurement from $\mathcal{D}_{\mathcal{A}}$. Precisely,

$$\mathcal{Q}(\mathcal{E}) = \inf_{\mathcal{D}_{\mathcal{A}}} \sum_{x=1}^N p_x H(\{p_{i|x}\}_{i=1}^n).$$

Here $p_{i|x}$ is the probability that η_i clicks if the signal state was ϱ_x, i.e.,

$$p_{i|x} = \mathrm{tr}(\eta_i \varrho_x).$$

The infimum is taken over all bases in $\mathcal{D}_{\mathcal{A}}$. Since the compactness of $\mathcal{D}_{\mathcal{A}}$ is not known, we keep the infimum instead of a minimum.

Actually, the above notion of $\mathcal{Q}$ as a measure of quantumness can be used in much more general situations than just in the case of a system consisting of two spatially localized subsystems. Suppose that a source produces an ensemble $\Gamma = \{p_x, \varrho_x\}_{x=1}^N$, with the ϱ_x's defined on some Hilbert space $\mathcal{H}$. And as before, let $\mathcal{A}$ denote the span of the supports of ϱ_x, and let $\{\eta_i\}_{i=1}^n$ be an arbitrary measurement in $\mathcal{H}$, whose rank-vector is majorized by that of the ensemble states. Let $\mathcal{D}_{\mathcal{A}}^{\Lambda}$ be the set of all measurements $\{\eta_i\}_{i=1}^n$, for which the nonzero overlap with $\mathcal{A}$ is distinguishable under a set chosen set of operations Λ. Then the measure of quantumness of the ensemble Γ, with respect to the set of allowed operations Λ, is defined as

$$\mathcal{Q}(\Gamma) = \inf_{\mathcal{D}_{\mathcal{A}}^{\Lambda}} \sum_{x=1}^N p_x H(\{p_{i|x}\}_{i=1}^n).$$

When the states ϱ_x are bipartite states on $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, with the allowed operations being LOCC, we recover the definition given before for bipartite ensembles.

For a multiparty ensemble $\{p_x, \varrho_x^{ABC\cdots}\}$, defined on $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \cdots$, one chooses the measurements $\{\eta_i^{ABC\cdots}\}$ on this Hilbert space. Considering the set of allowed operations as LOCC between $A, B, C, \ldots$, and $\mathcal{D}_{\mathcal{A}}$ as the set of measurements whose nonzero overlap with $\mathcal{A}$ is distinguishable by such LOCC operations, where $\mathcal{A}$ is the span of the supports of $\varrho_x^{ABC\cdots}$, we have a measure of quantum correlation of an ensemble of multiparty states.

Importantly, $\mathcal{Q}$ can be defined also for an ensemble Γ of a single system, when the allowed operations in its definition are all quantum operations. The dephasing is then in any orthonormal measurement (in $\mathcal{H}$), whose rank-vector is majorized by that of the ensemble states. Again the idea is to consider an ensemble consisting of the elements of such a measurement as "nonquantum." Any other ensemble will be "incommensurable" with all such nonquantum ensembles, during a measurement. And the entropy produced can again be interpreted as a measure of quantumness of Γ. The measure of quantumness for single systems proposed in [25] is based on "fidelity," while the one here, when considered for a single system, is based on entropy production. Note that Ref. [25] proposed that the "most quantum" two elements ensemble should be

$$\Gamma(\theta)=\{\cos(\theta/2)|0\rangle \pm \sin(\theta/2)|1\rangle\},$$

for $\theta=\pi/2$, and when given with equal probabilities. This criterion is satisfied by our measure, just as for the measure in Ref. [25]. In general,

$$\mathcal{Q}(\Gamma(\theta))=H(\cos^2(\theta/2)), \tag{3}$$

where we have assumed that the elements of $\Gamma(\theta)$ are given with equal probabilities. Here in the case of a single system, we assume that all quantum mechanical operations are allowed. Moreover, the ensemble states are pure, and span a two-dimensional complex Hilbert space ($\mathbb{C}^2$). Therefore, in this case, the set $\mathcal{D}_{\mathcal{A}}^{\Lambda}$ consists of all rank-one projective measurements in $\mathbb{C}^2$. Equation (3) then follows by direct calculation.

It may later on be reasonable to consider a regularized version of $\mathcal{Q}$. Given an ensemble $\{p_x,\varrho_x\}$ and a set of allowed operations Λ, consider $\mathcal{Q}'$ for an extension

$$\{p_x,\varrho_x\otimes|0\rangle\langle 0|\}$$

of the given ensemble, where $|0\rangle$ is any state that can be produced by Λ (hence it is for free). The regularization is then the infimum of $\mathcal{Q}'$'s for all possible such extensions. This brings in the possibility of vanishing regularized $\mathcal{Q}$ even for some ensembles that are indistinguishable under Λ. Unless mentioned otherwise, we will be considering the nonregularized version here.

By definition, $\mathcal{Q}$ is vanishing for ensembles which are distinguishable under the allowed set of operations Λ. For indistinguishable ensembles, due to possible noncompactness of the set over which the infimums are taken, $\mathcal{Q}$ can possibly be also vanishing.

IV. RELATION WITH RELATIVE ENTROPY DISTANCE

We now show that for pure ensembles, $\mathcal{Q}(\Gamma)$ is average relative entropy distance from some ensembles of states, where

$$S(\varrho|\sigma)=\mathrm{tr}(\varrho\log_2\varrho-\varrho\log_2\sigma)$$

is the relative entropy distance of ϱ from σ. Let $\{|a_i\rangle\}_i$ be any set of states which are distinguishable under the set of allowable operations Λ, and $\{a_i^x\}_i$'s be arbitrary sets of probabilities mixing the $|a_i\rangle$'s. Also, let

$$\sigma_x'=\sum_i p_{i|x}|a_i\rangle\langle a_i|$$

and

$$\sigma_x=\sum_i a_i^x|a_i\rangle\langle a_i|.$$

Then, from the definition of $\mathcal{Q}$, it follows that

$$\mathcal{Q}(\Gamma)=\inf_{\{\sigma_x'\}}\sum_x p_x[-\mathrm{tr}(\varrho_x\log_2\sigma_x')]=\inf_{\{\sigma_x'\}}\sum_x p_x S(\varrho_x|\sigma_x'),$$

so that

$$\mathcal{Q}(\Gamma)=\inf_{\{\sigma_x\}}\sum_x p_x S(\varrho_x|\sigma_x). \tag{4}$$

So for pure ensembles, $\mathcal{Q}$ is the average relative entropy distance of the signal states from the "classical" ensembles, namely, the ones whose members are mixtures of the same set of distinguishable pure states (distinguishable under Λ). Note that for a single "classical" ensemble, the constituent signals are mutually commuting. Although $\mathcal{Q}$ turns out to be a relative entropy distance, it has (in contrast to relative entropy of entanglement) an operational meaning in terms of entropy production. Thus $\mathcal{Q}$ is more related to quantum correlation of states in Ref. [26], than to relative entropy of entanglement. Note that the averaging in Eq. (4) is done before the infimum, and so the measure $\mathcal{Q}$ can still "feel" the ensemble as a whole. Otherwise, the measure will turn out to be an average of an individual property of the ensemble states, as noted in the Introduction.

Consider now the case of an ensemble of bipartite pure states with the allowed operations being LOCC between the sharing partners. Suppose also that the ensemble spans the whole bipartite Hilbert space. A set of orthogonal pure states in the span (i.e., a complete orthogonal basis) is distinguishable under LOCC only if they are all product states [21]. (Note that the opposite is not necessarily true [19].) So the relevant σ_x's that appear in formula (4) must have product eigen basis, and so are separable states. (Again the opposite is not true.) So

$$\mathcal{Q}(\mathcal{E})\geqslant\overline{E_R}\equiv\sum_x p_x E_R(\varrho_x^{AB})=\sum_x p_x S(\mathrm{tr}_A\varrho_x^{AB}), \tag{5}$$

for any pure ensemble $\mathcal{E}$ spanning the whole bipartite Hilbert space.

$$E_R(\varrho^{AB})=\min_{\sigma} S(\varrho|\sigma)$$

is the relative entropy of entanglement, the minimization being over all separable states σ [2]. We have used that for pure bipartite states

$$E_R(\varrho_x^{AB})=S(\mathrm{tr}_A\,\varrho_x^{AB}),$$

the unique asymptotic pure state entanglement measure [1,2]. Among other things, the inequality (5) will be important for evaluation of $\mathcal{Q}$ for certain bipartite ensembles.

V. EVALUATING THE MEASURE

We will now calculate the value of our measure for the ensemble $\mathcal{B}$ consisting of the four Bell states

$$|\phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle),$$

$$|\psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle),$$

given with equal prior probabilities. The four Bell states span the whole $2\otimes 2$ Hilbert space. To find $\mathcal{Q}$, we have to minimize entropy production after dephasing over all LOCC-distinguishable bases in $2\otimes 2$. Let us dephase in the computational basis

$$\{|00\rangle,|11\rangle,|01\rangle,|10\rangle\}.$$

For any signal, the entropy produced is $H(\frac{1}{2})$. So

$$\mathcal{Q} \leq 4 \times \frac{1}{4} H\left(\frac{1}{2}\right) = 1.$$

However from Eq. (5) we know that this bound is saturated, as the Bell states have $E_R=1$. Therefore, we have that

$$\mathcal{Q} = 1,$$

for the ensemble of the four Bell states given with equal prior probabilities. Consider now the more general ensemble $\mathcal{B}'$ consisting of

$$a|00\rangle + b|11\rangle, \quad \bar{b}|00\rangle - \bar{a}|11\rangle,$$

$$c|01\rangle + d|10\rangle, \quad \bar{d}|01\rangle - \bar{c}|10\rangle,$$

given with equal prior probabilities. Here a and b (c and d) are complex numbers, with $|a|^2+|b|^2=1$ ($|c|^2+|d|^2=1$). Again a dephasing in the computational basis implies that

$$\mathcal{Q} \leq \frac{1}{2}[H(|a|^2) + H(|c|^2)].$$

And again saturation follows from Eq. (5), so that

$$\mathcal{Q} = \frac{1}{2}[H(|a|^2) + H(|c|^2)].$$

Similarly, for the canonical set of maximally entangled states in $d\otimes d$, given with equal prior probabilities, viz.,

$$|\psi_{nm}^{\max}\rangle = \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1} e^{2\pi i j n/d}|j\rangle|(j+m)\mathrm{mod}\ d\rangle,$$

where $n,m=0,\ldots,d-1$, one obtains

$$\mathcal{Q} = \log_2 d,$$

by dephasing of this basis in the computational basis and using Eq. (5).

VI. MONOTONICITY AXIOMS

It should be noted that $\mathcal{Q}$, for the case of multipartite ensembles (with the allowed operations being the LOCC class) can actually increase under LOCC. This is because one can create nonorthogonal product states after starting with a multiorthogonal product basis [27]. Note, however, that $\mathcal{Q}$ can increase even if the output states are orthogonal, as there exist ensembles of orthogonal *product* states which are locally indistinguishable [19]. Such ensembles, which by definition have nonzero $\mathcal{Q}$, can be created by LOCC from a multi-orthogonal product basis [27], which being LOCC distinguishable has by definition vanishing $\mathcal{Q}$. However, from our experience with entanglementlike quantum correlation measures [1–4], we know that a quantum correlation measure should show some kind of monotonicity. It may now seem that $\mathcal{Q}$ (for the case of multipartite ensembles), being defined in terms of entropy production, will be monotonically decreasing under LOCC operations, if we do not allow discarding subsystems as a valid operation, because discarding subsystems can both increase or decrease entropy. Note that the operations that are useful in a distinguishing protocol of a multiparty ensemble are (a) adding local ancillas, (b) local unitarities, (c) local dephasing (von Neumann measurement), (d) communication of predephased quantum states (classical communication). Adding the item (e) discarding subsystems, will give us the whole set of LOCC. $\mathcal{Q}$ is obviously monotonically decreasing for operations (a), (b), and (d). However, it turns out that $\mathcal{Q}$ increases under (c): The ensemble

$$\mathcal{E}_1 = \{|s_{ij}^{AB}\rangle|ij\rangle^{A'B'}\}_{i,j=1}^{3},$$

where $\{|s_{ij}^{AB}\rangle\}_{i,j=1}^{3}$ is a set of nine orthogonal product states in $\mathbb{C}^3\otimes\mathbb{C}^3$ which are locally indistinguishable [19], and $\{|i\rangle^{A'}\}_{i=1}^{3}$, $\{|j\rangle^{B'}\}_{j=1}^{3}$ are local orthogonal states, can be transformed into $\{|s_{ij}^{AB}\rangle\}_{i,j=1}^{3}$ along with just white noise at $A'B'$, by performing a measurement in a basis complementary to $\{|ij\rangle^{A'B'}\}_{i,j=1}^{3}$, and forgetting the outcome. Since

$$\mathcal{Q}(\mathcal{E}_1) = 0$$

and

$$\mathcal{Q}(\{|s_{ij}\rangle\}) > 0,$$

we have an example where $\mathcal{Q}$ increases under (c).

Even if the outcomes in (c) are remembered, they are random and completely uncorrelated with anything else in this example, and so remembering them does not help. However, the preceeding discussion seems to indicate that monotonicity under addition of ancillas, unitary operations, and dephasing with outputs being orthogonal states, are natural axioms for any measure of quantumness of a single system.

VII. CONJECTURE: COMPLEMENTARITY WITH ACCESSIBLE INFORMATION

A. Formulation

Quantification of quantum correlation of an ensemble has several important potential applications. For example, it suggests a complementarity relation with locally accessible information (Sec. II). For an ensemble $\{p_x,\varrho_x^{AB}\}$, of bipartite states, a complementarity has been obtained [16] between the locally accessible information and average shared asymptotic entanglement E [4] of the ensemble states ($k=\dim\{\mathcal{H}_A \otimes \mathcal{H}_B\}$):

$$I_{\mathrm{acc}}^{\mathrm{LOCC}} + \bar{E} \leqslant \log_2 k. \tag{6}$$

However, the complexity of locally accessible information depends on the geometry of the ensemble, which cannot be captured by taking an average of an individual property (for example, entanglement) of the ensemble states. Consequently, the complementarity obtained in Ref. [16] can potentially be made stronger if the locally accessible information is taken along with a measure of quantum correlation of an ensemble of states.

Consider the quantity

$$I_{\mathrm{acc}}^{\Lambda} + \mathcal{Q},$$

where $I_{\mathrm{acc}}^{\Lambda}$ is the accessible information under the set of operations Λ (Sec. II), and $\mathcal{Q}$ is considered using the same set of operations. Below we prove that the complementarity relation between accessible information and quantumness, for an ensemble $\{p_x, \varrho_x\}_{x=1}^{N}$

$$I_{\mathrm{acc}}^{\Lambda} + \mathcal{Q} \leqslant \log_2 N \tag{7}$$

holds for several exemplary families. We feel that the complementarity (7) is true in general. Among other things, note that in the Eq. (6), the right-hand side (RHS) is the logarithm of the dimension of the Hilbert space on which the signals are defined, whereas in Eq. (7), the RHS is the logarithm of the number of states in the ensemble, which can be less than, equal to, or greater than the dimension of the said Hilbert space.

For an ensemble of bipartite states, $\{p_x, \varrho_x^{AB}\}_{x=1}^{N}$, the proposed complementarity is

$$I_{\mathrm{acc}}^{\mathrm{LOCC}} + \mathcal{Q} \leqslant \log_2 N. \tag{8}$$

Note that due to Eq. (5), the relation (8) will in general be stronger than the one in Eq. (6) if the ensemble consists of k pure states that span $\mathcal{H}_A \otimes \mathcal{H}_B$. It seems to be true that $\mathcal{Q} \geqslant \overline{E_R} + \log_2 N - \log_2 k$.

B. Exemplary cases

1. Some exemplary ensembles consisting of four states of two qubits

For the case of the four Bell states, given with equal prior probabilities $I_{\mathrm{acc}}^{\mathrm{LOCC}}=1$. This follows from Eq. (6) and the fact that measuring in the computational basis gives $I_{\mathrm{acc}}^{\mathrm{LOCC}} \geqslant 1$. And for this case, we have proven that $\mathcal{Q}=1$. Therefore we have proven inequality (8) for the case of four Bell states given with equal probabilities. Inequality (8) is true also for the more general ensemble $\mathcal{B}'$. From Eq. (6), we have $I_{\mathrm{acc}}^{\mathrm{LOCC}} \leqslant 2 - \frac{1}{2}[H(|a|^2) + H(|c|^2)]$. But a measurement in the computational basis shows that this bound can be achieved, so that in this case,

$$I_{\mathrm{acc}}^{\mathrm{LOCC}} = 2 - \frac{1}{2}[H(|a|^2) + H(|c|^2)].$$

Using the value of $\mathcal{Q}$ obtained for $\mathcal{B}'$, we have the inequality (8) proven for this ensemble. Similarly Eq. (8) holds for the canonical set of maximally entangled states in $d \otimes d$.

2. An ensemble of three Bell states

For the case of the ensemble of the three Bell states $|\phi^{\pm}\rangle$, $|\psi^{+}\rangle$, given with equal prior probabilities, a measurement in the computational basis gives $I_{\mathrm{acc}}^{\mathrm{LOCC}} \geqslant \log_2 3 - \frac{2}{3}$. And the fact that the set $\{|00\rangle, |11\rangle, |\psi^{+}\rangle\}$ is an LOCC-distinguishable ensemble belonging to the span of $|\phi^{\pm}\rangle$, $|\psi^{+}\rangle$, gives $\mathcal{Q} \leqslant \frac{2}{3}$. These two inequalities do not have a contradiction with the proposed Eq. (8).

VIII. STATUS OF THE CONJECTURED COMPLEMENTARITY FOR SINGLE SYSTEMS

One may consider Eq. (7) for the case of an ensemble of a single system, with all quantum mechanically allowed operations in Λ. Then Eq. (7) can be seen as a "Heisenberg uncertainty relation." The accessible information is the maximal "information gain" that is possible about the system. On the other hand, $\mathcal{Q}$ denotes the quantumness of the ensemble, quantifying the resistance to such information gain. For an ensemble $\{p_x, \varrho_x\}_{x=1}^{N}$ on $\mathcal{H}$, the accessible information

$$I_{\mathrm{acc}} = \max_{M} \left(H(\{p_x\}_{x=1}^{N}) - \sum_{y} r_y H(\{p_{x|y}\}_{x=1}^{N}) \right),$$

where a measurement $M=\{y\}$ has been performed, the measurement result y having occured with probability r_y. $p_{x|y}$ is the probability that the signal was ϱ_x, given that the outcome was y. Now

$$H(X) - H(X|Y) = H(Y) - H(Y|X),$$

for random variables X and Y. So

$$H(\{p_x\}_{x=1}^{N}) - \sum_{y} r_y H(\{p_{x|y}\}_{x=1}^{N}) = H(\{p_y\}_y) - \sum_{x=1}^{N} p_x H(\{p_{y|x}\}_y).$$

If we now restrict ourselves to only projection valued measurements on $\mathcal{H}$, we have

$$I_{\mathrm{acc}} \leqslant \max_{M} H(\{p_y\}_y) - \mathcal{Q} \leqslant \log_2 \dim \mathcal{H} - \mathcal{Q} \leqslant \log_2 N - \mathcal{Q}.$$

Hence for a single system, we have Eq. (7) in a restricted case. Let us mention here that it is conceivable that in general, the complementarity in Eq. (7) is true only when we consider the regularized version of $\mathcal{Q}$.

IX. CONCLUSION

We have proposed a measure of quantum correlation of ensembles. We evaluated the measure for several cases. We used it to propose a complementarity with accessible information. The measure and the complementarity are well-defined even for single systems, where the latter is proven when projection-valued measurements are used to maximize the accessible information. The measure naturally led to the first monotonicity axioms for quantumness of ensembles for single systems.

ACKNOWLEDGMENTS

We acknowledge support from the EU Grant No. RESQ, DFG (Grants No. SFB 407, No. SPP 1078, No. SPP 1116, and No. 436POL), the Alexander von Humboldt Foundation, Spanish MEC (Grant No. FIS-2005-04627, Consolider Project QOIT, and Ramón y Cajal), the ESF Program QUDEDIS, and EU IP SCALA.

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[23] $\eta_{i_k} \in \{\eta_i^{AB}\}_{i=1}^{n}$ has "nonzero overlap" with $\mathcal{A}$, if $\langle \wp | \eta_{i_k} | \wp \rangle \neq 0$, for some $|\wp\rangle \in \mathcal{A}$. The set of such η_{i_k} of $\{\eta_i^{AB}\}_{i=1}^{n}$ is called "overlap of $\{\eta_i^{AB}\}_{i=1}^{n}$ with $\mathcal{A}$."

[24] To be precise, a set of N states $\{\widetilde{\eta}_i\}$ are LOCC distinguishable if the corresponding ensemble of those states, given with equal probabilities, has an accessible information under LOCC equal to $\log_2 N$. If the accessible information under LOCC is strictly less than $\log_2 N$, then the set of states are LOCC-indistinguishable.

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[27] For orthogonal bases $\{|i\rangle\}_{i=1}^{\dim \mathcal{H}_k}$ of $\mathcal{H}_k (k=A,B,\ldots)$, $\{|i\rangle \otimes |j\rangle \otimes \cdots\}_{i,j,\cdots=1}^{\dim \mathcal{H}_A, \dim \mathcal{H}_B, \cdots}$ is the corresponding multiorthogonal basis. Such a basis is trivially distinguishable by LOCC between $A,B,\ldots$, and so has $\mathcal{Q}=0$.